Al-Kitab Journal for Pure Sciences
ISSN: 2617-1260 (print), 2617-8141(online)
https://isnra.net/index.php/kjps

Approximate Solutions of Nonlinear Integral Equations Using the Cubic B-Spline Scaling Method

Mohammed Jabbar Adaay Al-Sharea

Department of Mathematics, College of Science, University of Azad, Iran.

Corresponding Author: Mohammedalshara1975@gmail.com

Citation: AL-SHAREA MJA. Approximate Solutions of Nonlinear Integral Equations Using the Cubic B-Spline Scaling Method. Al-Kitab J. Pure Sci. [Internet]. 2025 Apr. 28;9(2):62-83. DOI: https://doi.org/10.32441/kjps.09.02.p5.

Keywords: Fixed Point Method, Non-Linear Fredholm Integral Equation, Cubic B-Spline Wavelets, Scaling Functions, Darbo Condition.

Received 31 Jul. 2024
Accepted 15 Oct. 2024
Available online 28 Apr. 2025

Abstract:

This paper examines a category of general nonlinear integral equations. These equations also include many special cases, such as functional equations and nonlinear integral equations of the Volterra type. In order to approximate the solutions to numerous physical, chemical, and biological issues, we implemented an approach that incorporates the fixed-point method and semi-vertical cubic scaling functions. We also obtain a numerical solution to the integral equation. Numerical examples illustrate the accuracy and validity of this method.

Keywords: Fixed Point Method, Non-Linear Fredholm Integral Equation, Cubic B-Spline Wavelets, Scaling Functions, Darbo Condition.

الحلول التقريبية للمعادلات التكاملية غير الخطية باستخدام طريقة قياس الشريحة *B* المكعبة

محمد جبار عداي الشرع

قسم الرياضيات/ التحليل العددي/ كلية العلوم/ جامعة آزاد الإسلامية/ إيران

Mohammedalshara1975@gmail.com

الخلاصة:

تتناول هذه الورقة فئة من المعادلات التكاملية غير الخطية العامة. تتضمن هذه المعادلات أيضًا العديد من الحالات الخاصة، مثل المعادلات الوظيفية والمعادلات التكاملية غير الخطية من نوع فولتيرا. من أجل تقريب الحلول للعديد من القضايا الفيزيائية والكيميائية والبيولوجية، قمنا بتنفيذ نهج يتضمن طريقة النقطة الثابتة ودوال التحجيم التكعيبي شبه العمودية، نحن نحصل أيضًا على حل عددي للمعادلة التكاملية. توضح الأمثلة العددية دقة وصلاحية هذه الطريقة.

الكلمات المفتاحية: طريقة النقطة الثابتة؛ معادلة فريدهولم التكاملية غير الخطية؛ الموجات المكعبة ذات شكل متعرج ؛ وظيفة القياس، شرط داربو.

1. Introduction:

An integral equation is an equation with an unknown function, x(s), under the integral sign **[1-4]**. The conventional form of this Equation in x(s) is as follows.

$$x(t) = f(t) + \lambda \int_{9(t)}^{f(t)} k(t.s)u(s)\,ds, \quad (1)$$

k(t,s) is a function that consists of two variables and is referred to as the kernel of the Equation, while λ is a constant parameter. The limits of integration are β(t) and g(t). The function x(s) appears, defined under the integral sign, as well as the interior and exterior of the sign. The functions f(t) and k(t,s) have been given previously, and the limits of integration g(t) and f(t) can be constants, variables, or a combination of a constant and a variable. Integral equations have multiple forms, and there are two ways to distinguish the Equation, which depend on the limits of integration.

1- If the limits of integration are constant, the Equation is referred to as the Fredholm equation and is expressed in the following representation.

$$x(t) = f(t) + \lambda \int_{a}^{b} k(t.s)u(s)\,ds \quad (2)$$

2- When one of the limits of integration is a constant, and the other is a variable, the equation is considered a Volterra equation and is expressed in the following manner.

$$x(t) = f(t) + \lambda \int_{a}^{x} k(t.s)u(s)\,ds, \quad (3)$$

Additionally, two varieties of equations are contingent upon the form of the function x(s), which is defined as follows:

1- The integral Equation is referred to as a Volterra or Fredholm equation of the first kind if the unknown function x(s) is present exclusively within the integral sign.

2- A Volterra or Fredholm equation of the second kind is defined as an equation in which The unknown function u(s) is present both within and outside the integral sign. Suppose the Function f(s) equals zero in Volterra or Fredholm equations. The integral Equation is referred to as a homogeneous equation. The Fredholm integral equation of the second kind is called nonlinear if the function x(s) that appears under the integral sign is nonlinear. Additionally, it is expressed in the subsequent manner ($e^{-cos^2(\frac{1}{2})}$. $sin^{2}(x)$…..), etc. Accordingly, the Equation has been formulated as follows.

$$x(t) = f(t) + \lambda \int_{9(t)}^{f(t)} k(t.s)F\big(u(s)\big)\,ds, \quad \textbf{(4)}$$

We want to clarify in this introduction that Fredholm's integral equations can be derived from boundary value problems, and it is essential to remember Eric Fredholm's work on integral equations and the applied theory from the year (1866-1927). The Swedish scientist developed the theory of integral equations, and his research paper, presented in 1903 in the Acta Mathematica, played a fundamental role in establishing operator theory. Integral equations play a prominent role in applied mathematics, and non-linear integral equations have significant practical importance, as shown by numerous studies in the field of knowledge, encompassing biology, traffic theory, optimal control theory, economics, and other engineering sciences **[5-8]**. Numerous sources have examined functional integral equations' existence and analytical behaviour **[5, 9, 10]** using non-compactness measure techniques and fixed-point theories. In references **[10]**, scholars Jalilian and Aghanjani presented numerous results related to the existence and unified universal gravity and the local gravity of solutions to the functional integral Equation.

$$x(t) = (Kx)(t) = f\left(t, x\big(\alpha(t)\big), \int_{0}^{\beta(t)} u\Big(t, s, x\big(\gamma(s)\big)\Big)\,ds\right), \qquad t \in [0, \infty] \quad \textbf{(5)}$$

These results were presented through the measure of non-compactness. These results were reached by the scientists Jalilian and Aghanjani, who worked to improve and expand upon the findings that emerged in other studies. Most functional integral equations are not amenable to analytical solutions; therefore, numerical methods are indispensable. Consequently, numerical methods are implemented to ascertain an Approximately calculated solution. The numerical solution of integral equations can be approached using projection, iterative, and Nystrom methods **[11-12]**; the references include the definitions of the collocation approaches **[13, 14-17]**. Galerkin methods are used to find numerical solutions for Fredholm integral equations, as outlined in the references **[13, 14, 17-21]** Spline functions, wavelets, product integration, homotopy analysis approaches, homotopy perturbation, Adomian decomposition method, interpolation of polynomials methods, suboptimal trajectories, and multigrid methods are all viable alternatives. The Nystrom procedures are mentioned in the references [11-14, 21, 22]. In a few articles, the approximate numerical representation of the solution has been analyzed. Composition techniques are the foundation of Numerical approaches to solving functional integral equations [23-30], homotopy perturbation methods [25,26-32], Lagrange and Chebyshev interpolation methods [27,28,32-38]. The various studies in most numerical methods addressed by previous research transform the integral Equation into a linear or non-linear algebraic equation system. This paper presents a numerical solution for an integral equation utilizing a hybrid iterative approach that combines the fixed-point method with trapezoidal scaling functions. The method does not rely on any equation-solving system. The objective of employing this method for non-linear functional integral equations is to achieve a more precise solution with less error. We successfully attained favorable outcomes with this strategy and further elaborated on the findings presented in other investigations. The investigation is structured as follows: In the first section, the scholars Jalilian and Aghanjani provide an introduction to integral equations and a definition of equation number (5). The second section of the research provides the definitions necessary for effectively composing this scientific paper. In the third section, we examined several findings that pertain to having existed and the allure of the aforementioned integral Equation. In the fourth section, we introduced the development of cubic B-spline functions within the interval [0.1], as documented in sources [39-42]. In the fifth section, we provide an explanation of the strategy used in this research's solution method and the method by which we approach the genuine solution. Conversely, in the final section, we provided numerical examples to demonstrate the precision of this methodology and contrasted the accuracy of these numerical results with those from prior research.

2. Background Concepts

We provide some definitions and findings in this section relevant to the rest of the paper. $BC(\mathcal{R}_+)$ is a branch with limited space, but operations continue on $\mathcal{R}_+$, furnished with a conventional standard. $||z|| = \sup\{|z(t)| : t \in \mathcal{R}_+, \}$.Let E Let be an infinite-dimensional Banach space containing the zero element. element θ and norm || · || Indicate the closed ball with radius r and center at x by writing B(z, r). Closure and convex property of Z, a nonempty subset of E, are shown by the symbols Z and Conv Z, respectively. Let also m_E be all relatively compact combinations: their family and n_E denote the family of all nonempty bounded subsets of E. We employ the concept found in [4] for the non-compactness metric.

Definition 1. When a mapping v: $m_E \to \mathcal{R}_+$ meets specific requirements, it can indicate non-compactness in E.

1. a family ker u = {X ∈ m_E : u(x) = 0} is not-empty and ker x ⊂ n_E.
2. X⊂ Y ⇒ v(X) ≤ v(Y).
3. u(Z) = u(Z).
4. u(Conv x) = u(x).
5. u(λX + (1 − λ)Y) ≤ λu(x) + (1 − λ)u(Y) for λ∈ [0, 1].
6. write an equation of closed originating from sets (Xn) from nE such that X_{N+1} ⊂ Xn(n = 1, 2, . . .) and if $\lim_{n\to\infty} x(X) = 0$ consequently, the intersection emerged. $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty. Given the Example of Banas and Goebel, we present a Darbo-type fixed-point theorem.[9]

Theorem (1): I will define E as a closed, convex, limited, and not-empty subset of the Banach space divided into sub-sets C.

Let L: E → E a constant was present. Mapping presupposes that there is a consistent. Z.∈ [0, 1) thus, u(F(X)) ≤ kw(X) before every non-empty sub-group of C that. Subsequently, L contains a fix-point within the set C. For any not empty bounding sub-set X of BC($\mathbb{R}_+$), x ∈ X, T >0 and ∈ ≥ 0 let

$$w^T(x,\epsilon) = \sup\{|x(s) - x(s)| : t, s \in [0,T], |t-s| \leq \in\}$$

$$w^T(x,\epsilon) = \sup\{u^T(x,\epsilon) : x \in X\},$$

$$w_0^T(x) = \lim_{€\to\infty} u_0^T(x,€),$$

$$w_0^T(x) = \lim_{T\to\infty} u_0^T(x),$$

X(t) = x(t) : the variables x ∈ X},

diam the function X(t) = sup{|x(t) − y(t)| : x, y ∈ X

And

$$\mu(X) = w_0\ (x) + \lim_{T\to\infty} \sup \text{ diam the function } X(t) \quad (6)$$

Banas has demonstrated in [43] that the function μ measures non-compactness via space. BE($\mathcal{R}_+$). The solution to the equation as operative from the BE($\mathcal{R}_+$) included in BE ($\mathcal{R}_+$)

$$(F\,x)(t) = x(t) \quad (7)$$

We will discuss the introduction: the attractiveness to Eq. (7).

Definition 2: [9] If a ball B(x_0, r) exists in space, then solutions to Eq. (7) are locally attractive. BC($\mathbb{R}_+$) so that for Any two solutions that are arbitrary to Eq. (7) that are part of B(x_0, r) ∩Ω that is in our possession.

$$\lim_{t\to\infty} \big(x(t) - y(t)\big) = 0 \quad \textbf{(8)}$$

Resolving the Eq. (7) is locally alluring at a uniform rate (or, similarly, asymptotic stability) if the limit (4) is consistent concerning $B(x_0,r) \cap \Omega$,

Theorem 2: There is a minimum of one solution to Eq. (5) in BC($\mathcal{R}_+$) for all of (5)'s solutions that have uniform local attraction.

Proof: This section presents a summary of the proof necessary for the subsequent sections. Refer to **[10]** for additional information.

First, the authors designed operator K in **[2]**; hence, for any x ∈ BC(R+)

$$(\mathrm{H}x)(t) = f\left(t, x\big(\alpha(t)\big), \int_0^{\beta(t)} u\left(t, s, x\big(\gamma(s)\big)\right) ds\right),$$

(H$\boldsymbol{x}$) is clearly continuous on $\mathbb{R}_+$.Next, for each arbitrarily fixed t ∈ $\mathbb{R}_+$

$$|(\mathrm{M}x)(t)| \le n\,|x(\alpha(t))| + M_0,$$

Where $M_0 = \sup\{|f(t, 0, 0)| : t \in R+\} + \Psi(2D),$

$$r = \frac{M_0}{1-n}.$$

Subsequently, they demonstrated that given A set that is not vacant X ⊂ (Br), μ(HX) < nμ(X), with n ∈ U(0,1). Accordingly, the Eq. (1) of functional integration is at least one solution in British Columbia. ($\boldsymbol{\mathcal{R}}$+), and(x) has a fixed- -point in (Br) for the operator H. based on Theorem 1. Every solution to Eq. (5) contained within the A member of the Ker μ group is ball Br

Corollary 1. If f (t, s, 0) is constrained by additional Constraints 1–4, then the solutions to the integration of the formula (5) are as follows: are generally attractive, as stated in Theory 2. Sufficient evidence. See **[27]**.

3- Cubic B-Aspline Scaling and Wavelet Function on [0.1]

In L2(R), you can utilize scaling functions to increase the size of any function. Extending these functions outside the integration domain is possible because they are specified across the natural line. This article considers B-spline scaling functions with compact support built for the bounded interval [0,1]. When using order m semi-orthogonal B-spline scaling functions, the requirement.

$$2^{J_0} \ge g$$

Has to be met for there to be one full inner scaling function. $\omega_{4,k}^{(j)}(x)$ represents these scaling functions. We'll employ scaling functions for cubic B-splines (cardinal B-splines of order g = 4). As a cubic spline, its scaling is denoted.

$$A_4(x) = \omega_4(x) = \begin{cases} \frac{1}{6}x^3 & 0 \le x < 1 \\ \frac{1}{6}(-3x^3 + 12x^2 - 12x + 24) & 1 \le x < 2 \\ \frac{1}{6}(3x^3 - 12x^2 + 60x - 44) & 2 \le x < 3 \\ \frac{1}{6}(4-x)^3 & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, the shape of its two-scale relation is

$$\omega_4(x) = \frac{1}{8}\omega_4(2x) + \frac{4}{8}\omega_4(2x-1) + \frac{6}{8}\omega_4(2x-2) + \frac{4}{8}\omega_4(2x-3) + \frac{1}{8}\omega_4(2x-4)$$

$$\omega_{4,-3}^{(3)}(x) = \begin{cases} (1-8x)^3 & 0 \le x < \frac{1}{8} \\ 0 & \text{otherwise} \end{cases}$$

With boundary scale, for example, the scaling factors used for j_0= j = 3 and m = 4 are enumerated below.

$$\omega_{4,-2}^{(3)}(x) = \begin{cases} 896x^3 - 288x^2 + 24x & 0 \le x < \frac{1}{8} \\ 2(1-4x)^3 & \frac{1}{8} \le x < \frac{2}{8} \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_{4,-1}^{(3)}(x) = \begin{cases} -\frac{1408}{3}x^3 + 96x^2 & 0 \le x < \frac{1}{8} \\ \frac{896}{3}x^3 - \frac{576}{3}x^2 + 36x - \frac{3}{2} & \frac{1}{8} \le x < \frac{2}{8} \\ -\frac{1}{6}(4x-3)^3 & \frac{2}{8} \le x < \frac{3}{8} \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_{4,-1}^{(j)}(x) = \omega_{4,k}^{(3)}(2^{j-3}x), k = -3, -2, -1 \quad j = 3,4 \dots \dots \dots$$

$$\omega_{4,5}^{(3)}(x) = \omega_{4,-1}^{(3)}(1-x),$$

$$\omega_{4,6}^{(3)}(x) = \omega_{4,-2}^{(3)}(1-x)$$

$$\omega_{4,7}^{(3)}(x) = \omega_{4,-3}^{(3)}(1-x)$$

$$\omega_{4,k}^{(j)}(x) = \omega_{4,2^j-k-4}^{(3)}\left(1 - 2^{j-3}x\right), \quad k = 2^j - 3, \dots \dots, 2^j - 1, \quad j = 3,4 \dots \dots \dots$$

Inner scaling:

$$\omega_{4,0}^{(3)}(x) = \begin{cases} \frac{256}{3}x^3 & 0 \leq x < \frac{1}{8} \\ -256x^3 - 128x^2 - 16x + \frac{2}{3} & \frac{1}{8} \leq x < \frac{2}{8} \\ 256x^3 - 256x^2 + 80x - \frac{22}{3} & \frac{2}{8} \leq x < \frac{3}{8} \\ \frac{32}{3}(1-2x)^3 & \frac{3}{8} \leq x < \frac{1}{2} \\ 0 & \text{Otherwise} \end{cases}$$

$$\omega_{4,1}^{(3)}(x) = \begin{cases} \frac{1}{6}(8x-1)^3 & \frac{1}{8} \leq x < \frac{2}{8} \\ -256x^3 + 224x^2 - 60x + \frac{31}{6} & \frac{2}{8} \leq x < \frac{3}{8} \\ 256x^3 - 352x^2 + 156x - \frac{131}{6} & \frac{3}{8} \leq x < \frac{1}{2} \\ \frac{1}{6}(5-8x)^3 & \frac{1}{2} \leq x < \frac{5}{8} \\ 0 & \text{Otherwise} \end{cases}$$

$$\omega_{4,2}^{(3)}(x) = \begin{cases} \frac{4}{3}(4x-1)^3 & \frac{2}{8} \leq x < \frac{3}{8} \\ -256x^3 + 320x^2 - 128x + \frac{50}{3} & \frac{3}{8} \leq x < \frac{1}{2} \\ 256x^3 - 448x^2 + 256x - \frac{142}{3} & \frac{1}{2} \leq x < \frac{5}{8} \\ \frac{1}{6}(6-8x)^3 & \frac{5}{8} \leq x < \frac{3}{4} \\ 0 & \text{Otherwise} \end{cases}$$

$$\omega_{4,3}^{(3)}(x) = \begin{cases} \frac{1}{6}(8x-3)^3 & \frac{3}{8} \leq x < \frac{1}{2} \\ -256x^3 + 416x^2 - 220x + \frac{229}{6} & \frac{1}{2} \leq x < \frac{5}{8} \\ 256x^3 - 544x^2 + 380x - \frac{521}{6} & \frac{5}{8} \leq x < \frac{3}{4} \\ \frac{1}{6}(7-8x)^3 & \frac{3}{4} \leq x < \frac{7}{8} \\ 0 & \text{Otherwise} \end{cases}$$

$$\omega_{4,4}^{(3)}(x) = \begin{cases} \frac{1}{6}(8x-4)^3 & \frac{1}{2} \leq x < \frac{5}{8} \\ -256x^3 + 416x^2 - 220x + \frac{229}{6} & \frac{5}{8} \leq x < \frac{3}{4} \\ 256x^3 - 544x^2 + 380x - \frac{521}{6} & \frac{3}{4} \leq x < \frac{7}{8} \\ \frac{1}{6}(7-8x)^3 & \frac{7}{8} \leq x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

$\omega_{4,k}^{(j)}(x) = \omega_{4,k}^{(3)}(2^{j-3}x - k), \quad k = 0,1, \ldots \ldots . 2^j - 4 \quad j = 3,4 \ldots \ldots \ldots$

$$\psi_4(x) = \frac{1}{8!}\omega_4(2x) + \frac{124}{8!}\omega_4(2x-1) + \frac{1677}{8!}\omega_4(2x-2) + \frac{7904}{8!}\omega_4(2x-3) + \frac{18482}{8!}\omega_4(2x-4) - \frac{24264}{8!}\omega_4(2x-5) + \frac{18482}{8!}\omega_4(2x-6) - \frac{7904}{8!}\omega_4(2x-7) + \frac{1677}{8!}\omega_4(2x-8) - \frac{124}{8!}\omega_4(2x-9) + \frac{1}{8!}\omega_4(2x-10)$$

Cubic B-spline wavelet $\psi_4(x)$ is shown in Fig.3. The system's inner and border wavelet analysis is obtained through the application of [8, 11].

4. Functional Approximation

A function f (x) specified in the interval [0,1] can be rendered in the cubic B-spline scale field FJ_0

For any fixed positive integer J_0 as

$$g(x) = \sum_{i=-3}^{2^{j_0-1}} c_{j_0.i}\varphi_{4.i}^{(3\ )}(x) + \sum_{i=3}^{j_0}\sum_{k=-3}^{2^{i}-4} d_{i.k}\Psi_{4.k}^{(i)}(x) = C^T\Psi(x) \quad (9)$$

Where $\varphi_{4.i}^{(j_0)}$ and $\Psi_{4.k}^{(i)}$are wavelet and scaling functions should eq (9)'s infinite series be shortened, respectively. And for j=3, it can be expressed as follows:

$$g(x) = \sum_{i=-3}^{7} c_{j_0.i}\varphi_{4.i}^{(j_0)}(x) + \sum_{i-3}^{j_0}\sum_{k=-3}^{2^{i}-4} d_{i.k}\Psi_{4.k}^{(i)}(x) = C^T\Psi(x) \quad (10)$$

Where C and $\Psi(x)$ are $(2^{j_0+1}+5)$x1 vectors given bye

$$C = "[c_{-3,\ldots\ldots}, c_{7,,} d_{3,-3,\ldots\ldots} d_{3.4\ldots\ldots} d_{j0,-3\ \ldots\ldots} d_{j2^{j}-4\ ,]}.^{T}" \quad (11)$$

$$\Psi = [\varphi_{4.-3}^{(3)} \ldots\ldots . \varphi_{4.-3}^{(3)}, \Psi_{4.-3}^{(3)} \ldots \Psi_{4.-3}^{(3)} \ldots \Psi_{4.4}^{(3)} \ldots\ldots \Psi_{j_0.2^{j}-3}^{(3)}].^{T} \quad (12)$$

With

$$C_1 = \int_0^1 f(x)\varphi_{4.1}^{(3)}(x)\,dx, \ldots . i = -3, \ldots\ldots 7. \quad (13)$$

$$d_{j.k} = \int_0^1 f(x)\psi_{4.1}^{(j)}(x)\,dx, \ldots . j = 3, \ldots J_0 . k = -3 \ldots\ldots 2^{J_0} - 4, \quad (14)$$

Where $\varphi_{4.1}^{(3)}$ and $\psi_{4.k}^{(j)}$ have dual purposes. of $\varphi_{4.1}^{(3)} i = -3, \ldots\ldots 7$ and $\psi_{4.k}^{(j)} = j = 3, \ldots J_0$ According to. By using linear combinations, these can be obtained. Of

$\varphi_{4.1}^{(3)}$ and $\psi_{4.k}^{(j)}$

$$\varphi = [\varphi_{4.-3}^{(3)}\ _{,}(x)\varphi_{4.-2}^{(3)}(x) \ldots\ldots . \varphi_{4.-7}^{(3)}(x)\ ].^{T} \quad (15)$$

$$\Psi = [\Psi_{4.-3}^{(3)}(x) \ldots \Psi_{4.4}^{(3)}(x) \ldots\ldots \Psi_{4.2^{j}-4}^{(3)}(x)].^{T} \quad (16)$$

Using(9)-(13),(15,16) We obtain

$$\int_0^1 \phi(x)\phi^T(x)\,dx = L_1 \qquad \textbf{(17)}$$

$$\begin{vmatrix}
\frac{1}{56} & \frac{7}{640} & \frac{31}{13440} & \frac{1}{566720} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7}{640} & \frac{31}{1120} & \frac{5}{256} & \frac{29}{6720} & \frac{1}{26880} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{31}{13440} & \frac{5}{256} & \frac{183}{4480} & \frac{283}{10080} & \frac{239}{80640} & \frac{1}{40320} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6720} & \frac{29}{6720} & \frac{283}{10080} & \frac{151}{2520} & \frac{397}{13440} & \frac{2}{672} & \frac{1}{40320} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{26880} & \frac{239}{80640} & \frac{397}{13440} & \frac{151}{2520} & \frac{397}{13440} & \frac{2}{672} & \frac{1}{40320} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{40320} & \frac{2}{672} & \frac{397}{13440} & \frac{151}{2520} & \frac{397}{13440} & \frac{2}{672} & \frac{1}{40320} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{40320} & \frac{2}{672} & \frac{397}{13440} & \frac{151}{2520} & \frac{283}{10080} & \frac{29}{6720} & \frac{1}{6720} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{40320} & \frac{2}{672} & \frac{397}{13440} & \frac{151}{2520} & \frac{283}{10080} & \frac{29}{6720} & \frac{1}{6720} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{40320} & \frac{239}{80640} & \frac{283}{10080} & \frac{183}{4480} & \frac{5}{256} & \frac{31}{13440} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{26880} & \frac{29}{6720} & \frac{5}{256} & \frac{31}{1120} & \frac{7}{640} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6720} & \frac{31}{13440} & \frac{7}{640} & \frac{1}{56}
\end{vmatrix}$$

L is [11 × 11].

Assume ϕ(x) serves dual functions. Of ϕ(x) as presented in the Equation

$\varphi = [\varphi^{(3)}_{4.-3}\ _{,}(x)\varphi^{(3)}_{4.-2}(x) \dots \dots .\, \varphi^{(3)}_{4.-7}(x)\ \ ].^T$

Using (8) (9) (11)

$$\int_0^1 \phi(x)\phi^T(x)\,dx = K11$$

where the identity matrix is [11 × 11]. Consequently, we obtain

$$\bar{\mathrm{w}} = \phi^{-1}\omega \qquad \textbf{(18)}$$

Theorem 3: Let $e_j(x)$ denote the approximate errors of f in (9) using cubic B-spline scaling function within space Vj; therefore, || = O$e_j(x)$ (2^{-4j}).

Proof. By using (9) and (10). We get

$$e_j(x) = \sum_{i=j}^{\infty} \sum_{k=-3}^{2^{i}-4} d_{i.k} \Psi_{4.k}^{(i)}(x)$$

By putting

$c_j = max\{|\psi^1(x)|; k = -3, \dots \dots 2^{i} - 4\}$ We obtain

And

$$\sum_{k=-3}^{2^{i}-4} \left| d_{i.k} \Psi_{4.k}^{(i)}(x) \right| \le \alpha\beta c_i \frac{2^{-4!}}{4!}$$

As a result,

$$\left| e_j(x) \right| \le \frac{1}{4!} \alpha\beta \sum_{i=j}^{\infty} c_i 2^{-4!}$$

The current inequality allows us to get

$$\left| e_j(x) \right| = O(2^{-4!}) \quad \textbf{(19)}$$

The order of error depends on the level j. as ()demonstrates. The approximation error will decrease with increasing degree of j.

5. Method of Solvation

This part outlines our primary approach, which combines the fixed point with the cub-spline scale function. Next, we consider the method's convergence.

5.1 The New Numerical Method's Description: When considering the integral Eq. (5). To streamline this procedure, assume that each value of t is confined to the interval where the maximum level of β(t) is constant. assume t ∈ [0, a] without losing generality. Allow me to

$$0 = t_0 \le t_1 \le t_m = a.,$$

By G locations in [0, a].

That are evenly separated. By the proof, K is a continuous operator on (Br) and had a fix-point x in the Br—theorem 2**.** Eq. (1) has at least one solution in Br under the assumptions 1–4. Additionally, evenly locally appealing solutions for problems (1). This section provides a concise overview of the evidence required in the subsequent sections. For additional information, please refer to **[6]**.

Initially, parameter G was defined by the creators in **[6]**. in a way that ensures that for any x ∈ Br,

$$x(t) = (\mathrm{K}x)(t) = f \cdot \left(t, x(\alpha(t)), \int_0^{\beta(t)} u\left(t, s, x\left(\gamma(s)\right)\right) ds\right) \ ,$$

Now, we treat operator H using the fix-point approach. for $\boldsymbol{x_0}$ (t) ∈ *Br* as well as Points t_I (i = 1, . . …….. ,G)

$$x_{k+1}(t_i) = (H_k)(t_i) = f \cdot \left(t_i, x_k(\alpha(t)), \int_0^{\beta(t)} u\left(t_i, s, x_k\left(\gamma(s)\right)\right) ds\right) K = 0,1,. \qquad \textbf{(20)}$$

The integral is to be approximated. Numerically inside the intervals [0, β(t_i)] in (21) and $x_{N+1}(t)$, we apply A Simpson rule that is composite and applies to equally distant L points. Today, we utilize cub-spline scaling methods as the foundation for our estimations. $x_{N+1}(t)$ to get ready for the following iteration. $x_{N+1}(t)$We can immediately calculate using the coefficients of the scaling functions (5)–(14) from the previous section without having to solve any systems of algebraic equations as

$z_i = \int_0^1 . x_{k+1}(t)\varpi_{4.J}^{(3)}(t)\, dt \ , J = -3, \dots . . 7$

where, as previously stated

$[\varpi_{4.-3}^{(3)}(t), \varpi_{4.-2}^{(3)}(t), \dots \dots . \varpi_{4.7}^{(3)}(t)]^T = P^{-1}\ [\omega_{4.-3}^{(3)}(t), \dots \dots . \omega_{4.7}^{(3)}(t)]^T$

Given the values of xk+1(ti) (i = 1,..., H), we compute cj. By employing the composite Simpson rule, we arrive at the following;

$$x_{k+1}(t) \approx \sum_{i=-3}^{7} c_{j.}\varphi_{4.}^{(3)}(t) \qquad \textbf{(21)}$$

We repeat the iterations until the difference between subsequent iterations, $x_k(t)$, is as small as we need for the appropriate level of precision. The end values of $x_{k+1}(t)$ correspond to an operator's M fixed point k at that level of precision. Consequently, we make an approximation of Equation (5). The following briefly describes the numerical approach.

5.2 There is a Relationship Between Teachers K and G: The teacher k represents the number of iterations within the fixed-point method, while G, according to the assumptions of Theory (2), are distant and central points within the interval [0. a]. Through the practical application of numerical examples, we continue with the iterations until we achieve a small difference between consecutive iterations. These small differences are essential for achieving high accuracy, ensuring that the terminal parameters x(t) converge to a stationary point for operator Z. Therefore, we are approaching an accurate solution to the Eq. (5). We observe the accuracy of the method used through the numerical examples in examples **[7]** and **[8]**, comparing them with the methods used in previous studies. We notice that an increase in k and G leads to an increase in accuracy and a decrease in the absolute error rate. The iterations keep increasing until the approximate solutions get closer to the exact solutions.

6- Results:

This article illustrates the method's accuracy by presenting numerical illustrations for the integral equation eq(5). We utilized the symbol k to represent the number of iterations in the fixed-point method. And K to Represent the approximate value x(t) in the iteration X(t), which is based. On this, we can calculate the absolute error ratio to x(t) in iteration k as follows: is represented with $x_{K(}t_{i)}$

$$|(X(t)) - x_{K+1}(t)|$$

Hence, it is possible to determine the most significant (absolute error in iteration K as an

$$\|x - x_K\| = max\left|x(t_{i)} - x_{K(}t_{i)}\right|$$

Furthermore, one can compute the discrepancy located between the approximate at K and K+1 as

$$\left|x_{K+1}(t) - x_{K(}t_{)}\right|.$$

As a result, we acquire

$$\|x - x_K\| = max\left|x_{K+1}(t) - x_{K(}t_{i)}\right|$$

We used different values of k and G to solve the following numerical examples. In these examples, we applied the formulas in the articles above (17) and (18) to derive approximate numerical solutions. The calculations and results were carried out using Mathematics 8.

Example (1) : non-linear Fredholm–Hammerstein equation that follows

$$x(t) = sin\left(\frac{\pi}{2}t\right) - 2te^{-t}ln(3) + e^{-t}\int_0^1 .\frac{4ts + \pi tsin(\pi s)}{x(s)^2 + s^2 + 1}ds,$$

Table 1: Absolute errors for Example 1 across various values of K.

t	k=2	k=5	k=10
0	0	0	0
0.1	0.127890×10^{-1}	0.953012×10^{-5}	0.362567×10^{-9}
0.2	0.214093×10^{-1}	0.167951×10^{-4}	0.119137×10^{-8}
0.3	0.227909×10^{-1}	0.197670×10^{-4}	0.139588×10^{-7}
0.4	0.169766×10^{-1}	0.149812×10^{-4}	0.402300×10^{-7}
0.5	0.797177×10^{-2}	0.497886×10^{-4}	0.934621×10^{-7}
0.6	0.384727×10^{-4}	0.334333×10^{-5}	0.378213×10^{-7}
0.7	0.472688×10^{-2}	0.660849×10^{-5}	0.269398×10^{-7}
0.8	0.631157×10^{-2}	0.612657×10^{-5}	0.301620×10^{-7}
0.9	0.572703×10^{-2}	0.453392×10^{-5}	0.464820×10^{-7}
1	0.410323×10^{-2}	0.290023×10^{-5}	0.301723×10^{-8}

Possesses an exclusive, precise resolution

. $x(t) = sin\left(\frac{\pi}{2}t\right)$. Functional, the value of β(t) is 1, and γ(t) is equal to t.

$$f(t,x,y) = sin\left(\frac{\pi}{2}t\right) - 2te^{-t}ln(3) + e^{-t}y$$

And

$$u(t,s,x)\frac{4ts+\pi t\sin(\pi s)}{x^2+s^2+1},$$

Are continuous functions that meet Theorem 2's presumptions. see

ϕ(t) Equal t, k Equals 0, m(t) = e^(-t), where D = 1.24575

We make a decision $x_0(t) = \sin\left(\frac{\pi}{2}t\right) - 2te^{-t}\ln(3) \in [-r,r]$, where $r = \frac{M_0}{1-n}. = 3.82984$.

Table 1 displays the absolute values of defects for (G = 200 and L = 200) mesh points. The errors associated with a single iteration are represented by absolute values in **Table 2**. for varied numbers of mesh points. Furthermore, errors are diminished by conducting additional iterations. Refer to Figure 1. Where the [$\log10(\|x_k - x\|) = \log10(\max|x_k(t_i) - x(t_i)|)$, to t_i = i/10 for any (i = 0, 1…… , 10)

Example (2): Examine the subsequent information. Kind (42, 43) of non-linear function integral equation of the Volterra.

$$x(t) = \frac{t}{1+t^2}x(t) + \int_0^t e^{-t}\frac{sx(s)}{1+|x(s)|}ds$$

Table 2: Absolute Errors within the EX.1 K Equal 10, L Equal 200, many G variables

t	G=50	G=100	G=200
0	0	0	0
0.1	0.170303×10^{-5}	0.449863×10^{-5}	0.362567×10^{-9}
0.2	0.307753×10^{-5}	0.805820×10^{-5}	0.119137×10^{-8}
0.3	0.409215×10^{-5}	0.104431×10^{-4}	0.139588×10^{-7}
0.4	0.478569×10^{-5}	0.117460×10^{-4}	0.402300×10^{-7}
0.5	0.521326×10^{-4}	0.121565×10^{-4}	0.934621×10^{-7}
0.6	0.540250×10^{-4}	0.1200099×10^{-4}	0.378213×10^{-7}
0.7	0.548649×10^{-4}	0.116624×10^{-4}	0.269398×10^{-7}
0.8	0.547760×10^{-4}	0.110637×10^{-4}	0.301620×10^{-7}
0.9	0.542085×10^{-4}	0.105845×10^{-4}	0.464820×10^{-7}
1	0.532832×10^{-4}	0.101492×10^{-4}	0.301723×10^{-7}

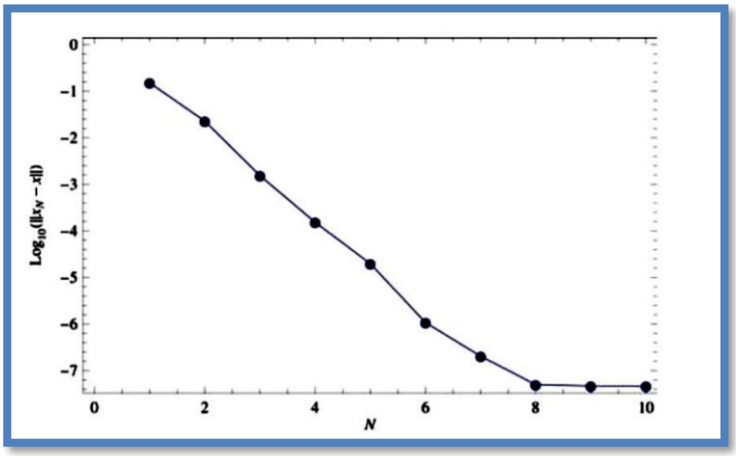

Figure 1: Ex(1) Illustrates the Logarithm of t the Utmost Error Occurring During Every Iteration.

Which has a unique, precise solution. The value of x(t) is zero. The formulas are: α(t) = β(t) = γ(t) = t: We can ascertain t by employing fundamental mathematics. the solution to which is unique and exact x(t) = 0. The functions α(t), β(t), and γ(t) are all defined as t.

$$f(t,s,x)\frac{tx}{1+t^2}+y,$$

And

$$u(t,s,x)=e^{-t}\frac{sx}{1+|x|}$$

Satisfying the requirements of the second theorem with ϕ(t) Equal t, k Equal 1/2, m(t) Equal 1, and D = 0.270671, The procedure is implemented. where x_0(t) Equal 0.5 ∈ Br, where r = 1.08268. **Table 3** displays the exact error values, while Figure 2 depicts the logarithm of the highest error associated with iterations, especially for G equals 80 and L equals 50 mesh points.

Example (3): We investigate the non-linear Volterra functional integral problem using proportional delay.

$$x(t)=\cos(t)-\sin\left(\frac{e^{-\cos^2(\frac{1}{2})}-e^{-1}}{1+t^2}\right)+\sin\left(\int_0^{t/2}\frac{\sin(2s)e^{-x^2(s)}}{1+t^2}ds\right),$$

Table3: Fundamental errors for Example 2 over several levels of K

t	k=5	k=25	k=50
0	0	0	0
0.1	0.462407×10^{-5}	0.808969×10^{-13}	0.306027×10^{-20}
0.2	0.131506×10^{-3}	0.268324×10^{-12}	0.107748×10^{-19}
0.3	0.790865×10^{-3}	0.275392×10^{-12}	0.171832×10^{-19}
0.4	0.243661×10^{-2}	0.300346×10^{-12}	0.114517×10^{-19}
0.5	0.512032×10^{-2}	0.649454×10^{-10}	0.156528×10^{-18}
0.6	$0.0.835707\times10^{-2}$	0.635856×10^{-9}	0.148931×10^{-17}
0.7	0.114419×10^{-1}	0.316263×10^{-8}	0.185272×10^{-16}
0.8	0.138099×10^{-1}	0.802217×10^{-8}	0.131274×10^{-15}
0.9	0.151984×10^{-1}	0.129801×10^{-7}	0.334904×10^{-15}
1	0.156251×10^{-1}	0.149012×10^{-7}	0.439281×10^{-15}

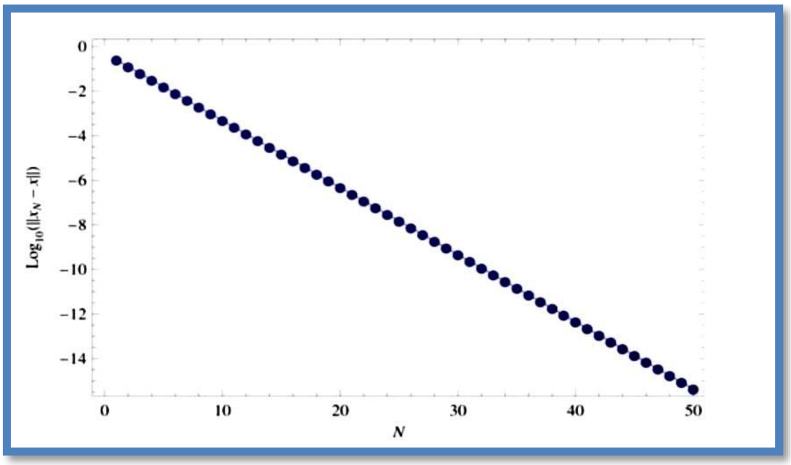

Figure 2: The logarithm of the highest error associated with every iteration in Example 2.

Which possesses the precise solutions x(t) = cos(t). The functions such as β(t) = t - (1 - q)t for q = 1/2 and γ(t) Equal t can be readily demonstrated to be continuous. Additionally, α(t) may be Any continuous operation that satisfies Condition 1.

$$f(t,x,y)=\cos(t) - \sin\left(\frac{e^{-\cos^2(\frac{1}{2})}-e^{-1}}{1+t^2}\right) + \sin(y)$$

And
$$u(t,s,x)=\cos(t)- \sin\left(\frac{e^{-x^2}\sin(2x)}{1+t^2}\right)$$

Meet the conditions of Theorem 2, given k = 0, m(t) = 1, ϕ(t) = t, with D ranging from 0.25. We

Derive r = 1.5 and utilize $x_0(t)= \cos(t)- \sin\left(\frac{e^{-\cos^2(\frac{1}{2})}-e^{-1}}{1+t^2}\right) \in Br$

The preliminary function. Both **Table 4** and the following figure illustrate the absolute errors with the logarithm of the highest Iteration-related absolute errors for L = 100 and G = 200 centered within the range [0,2].

Table 4: Approximate Errors for the Third Example for Various Values of K

t	k=2	k=5
0	0	0
0.2	0.234328×10^{-7}	0.239081×10^{-7}
0.4	0.146720×10^{-7}	0.360304×10^{-7}
0.6	0.323011×10^{-6}	0.914190×10^{-7}
0.8	0.863026×10^{-6}	0.637923×10^{-9}
1	0.205252×10^{-5}	0.284243×10^{-9}
1.2	0.539092×10^{-5}	0.878520×10^{-7}
1.4	0.113864×10^{-4}	0.593672×10^{-7}
1.6	0.154702×10^{-4}	0.506919×10^{-7}
1.8	0.252727×10^{-4}	0.231660×10^{-7}
2	0.852385×10^{-7}	0.116951×10^{-8}

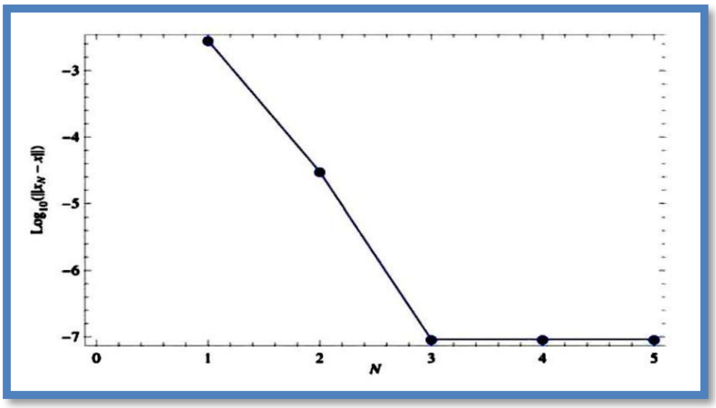

Figure 3. Logarithm of the greatest error associated with each repetition in Example (3).

Example (4): The subsequent non-linear integral Equation for Volterra function

$$x(t) = e^{-t\sqrt{t}} - \frac{t^2(1+e^{-t\sqrt{t}}x(t))}{2(1+t^4)} + \left(\frac{t^2}{1+t^4}\right) e^{-\left|\int_0^{t\sqrt{t}} \frac{2e^{-s}x(\sqrt[3]{s^2})}{(1+x(\sqrt[3]{s^2})^2)} ds\right|}$$

Contains a precise solution. The value of x(t) is equal to e. The functions to use are α(t) = t, β(t) = t, and γ(t) Equal 3.

Comply with the first condition of Theorem 2. Furthermore, functions as well

$$f(t,x,y) = e^{-t\sqrt{t}} - \frac{t^2(1+e^{-t\sqrt{tx}})}{2(1+t^4)} + \left(\frac{t^2}{1+t^4}\right)e^{-|y|}$$

And $$u(t,s,x) = \frac{2e^{-t}x}{1+x^2}$$

Fulfilled the criteria of Theorem 2–4 using n = 0.111641, ϕ(t) Equal t, $G(t) = t^2/(1+t^4)$, D = 0.5, and $M_0 = 2$. Thus, we have r = 2.25134 and choose $x_0(t) \in Br$. The extent of errors

Table 5: Absolute Errors of Example 4fore $x_0(t)$ =0.5and Different Value of K

t	K=2	K=5	K=10
0	0	0	0
0.1	0.116062×10^{-4}	0.302756×10^{-6}	0.302763×10^{-6}
0.2	0.150179×10^{-3}	0.164254×10^{-6}	0.163322×10^{-6}
0.3	0.597392×10^{-3}	0.518220×10^{-6}	0.551065×10^{-6}
0.4	0.139726×10^{-2}	0.963558×10^{-6}	0.626378×10^{-6}
0.5	0.237648×10^{-2}	0.135418×10^{-5}	0.213892×10^{-6}
0.6	0.319742×10^{-2}	0.444057×10^{-5}	0.933230×10^{-7}
0.7	0.356863×10^{-2}	0.811830×10^{-5}	0.411882×10^{-7}
0.8	0.342076×10^{-2}	0.110323×10^{-4}	0.286523×10^{-6}
0.9	0.291232×10^{-2}	0.133351×10^{-4}	0.152142×10^{-6}
1	0.22864×10^{-2}	0.136016×10^{-4}	0.216699×10^{-6}

Table 6:Absolute errors of Example 4for K=5 and different values of x_0(t)

t	$x_0(t)$=0.5	$x_0(t) = 1$	$x_0(t)$=2
0	0	0	0
0.1	0.302756×10^{-6}	0.302762×10^{-6}	0.302777×10^{-6}
0.2	0.164254×10^{-6}	0.163079×10^{-6}	0.161031×10^{-6}
0.3	0.518220×10^{-6}	0.562475×10^{-6}	0.637346×10^{-6}
0.4	0.963558×10^{-6}	0.420642×10^{-6}	0.419744×10^{-6}
0.5	0.135418×10^{-5}	0.165281×10^{-5}	0.595858×10^{-5}
0.6	0.444057×10^{-5}	0.553731×10^{-5}	0.188352×10^{-4}
0.7	0.811830×10^{-5}	0.140381×10^{-4}	0.417234×10^{-4}
0.8	0.110323×10^{-4}	0.251467×10^{-4}	0.680604×10^{-4}
0.9	0.133351×10^{-4}	0.133351×10^{-4}	0.880009×10^{-4}
1	0.136016×10^{-4}	0.397919×10^{-4}	0.991856×10^{-4}

Table 5 provides details of the iterations, which correspond to G = 150 and L = 100 mesh points. **Table 6** displays the maximum inaccuracy values for a single iteration according to various initial locations and a consistent number of mesh points. **Figure 4** illustrates how increasing the number of repetitions reduces mistakes.

Example (5): Examine the integral Equation shown below [10].

$$x(h) = \frac{\sin\left(hx(\sqrt[3]{h})\right)}{1+h^2} + \arctan\left(\int_o^{\sqrt{1}} \frac{\sqrt[4]{1+sx^2(s^2)+}}{(1+h^7)} \frac{hs^{11}(1+x^4(x^2))}{(1+x^4(x^2))}\right)ds.$$

Using basic math, we can determine that

$$\propto (h) = \sqrt[3]{t},\ \beta(h) = \sqrt{h},\ \text{y(t)}= h^2$$

$$x(h) = \frac{sin\left(hx(\sqrt[3]{h})\right)}{1+h^2}, +arctan(y)$$

And

$$u(t.s,x) = \frac{\sqrt[4]{1 + sx^2(s^2) + ts^{11}(1 + x^4)}}{(1 + h^7)(1 + x^4)}$$

Fulfill the requirements of Theory 2 accompanied by { D = 1.0184, k = 0.5, m(t) = 1, and ϕ(t) = h.} As a result, we determine that the beginning function belongs to the ball Br and get(r = 4.07362.) In contrast, since f (t, x, 0) is a function restricted in its range, Corollary 1's requirement are met. As a result, there is at least one solution to the integral Equation, and These solutions possess universal appeal. Upon completing 50 cycles, we ascertain the mistake for (L = 50 and G is equal to 80) mesh points.$\|x_{N+1} - x_N\|$ = maxi $|x_{N+1}(t_i) - x_N(t_i)|$ = 2.37155 × 10−8, where h_i = i/10 where i = 0, 1, ..., 10. In the figure, the response is an approximation obtained after 50 iterations. as demonstrated by the explanation that follows x(t) is an element of Br

Example (6): Examine: Kindly assess the presented non-linear Fredholm integral. [44,45]

$$x(t) = 1 + te - \int_o^1 (t + S)e^{X(S)}\, dS.$$

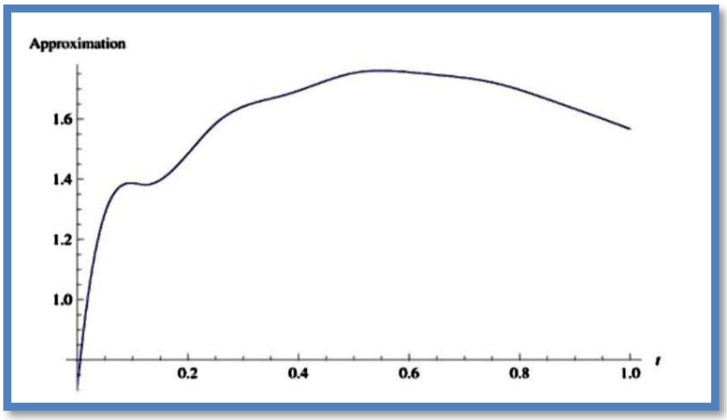

Figure 4: Approximation Solution of Example 5 after 50 Iteration

The precise solution is x(t) = t. This section addresses the solution of the integral Equation utilizing our suggested method and juxtaposes prior results with the new findings delineated in Table 7 for K = 30, G = 200, and L = 200.

Example (7): The following integral Equation

$$x(t) = sin(t)^2 + 1 - \int_0^t 3\, sin(t - s)x\,(s)^2\, ds$$

Table 7:As shown in Figure 3, the absolute errors are as follows: K=30, G=200, and D=200.

t	Methodology of[44]	Methodology [45]	Methodologically presented
0	2×10^{-3}	2.58×10^{-6}	8.96539×10^{-10}
0.2	1×10^{-2}	7.35×10^{-6}	7.99324×10^{-9}
0.4	2×10^{-2}	7.93×10^{-6}	$4.23015 \times 10^{--8}$
0.6	1×10^{-2}	2.55×10^{-6}	8.28779×10^{-9}
0.8	0×10^{-3}	3.98×10^{-6}	2.82284×10^{-7}
1	1×10^{-2}	2.64×10^{-6}	5.62541×10^{-8}

Table 8:Contains The Solutions to Example 7: K = 10, G = 250, And D = 200, With Both Exact And Approximate Values

t	Methodology of[44]	Methodology of[45]	Methodologically presented	PERFECT RESOLUTION
0	1.0000	1.00000	1.000000000	1.000000000
0.1	0.9952	0.99500	0.995004165	0.995004165
0.2	0.9800	0.98006	0.980066580	0.980066577
0.3	0.9554	0.95533	0.955336485	0.955336489
0.4	0.9210	0.92105	0.921060993	0.921060994
0.5	0.8775	0.87756	0.877582565	0.877582561
0.6	0.8255	0.82531	0.825335614	0.825335614
0.7	0.7648	0.76482	0.764842185	0.764842187
0.8	0.6969	0.69669	0.696706707	0.696706709
0.9	0.6217	0.62159	0.621609968	0.621609968
1	0.5405	0.54028	0.540302303	0.540302305

The exact answer is x(t) Equal cos(t). This Equation fails to satisfy the following criteria: to theory 2, we employ the technique described in this article. Furthermore, the findings of our investigation have been compared with the results **[46,47] Table 8**.

7. Conclusion

In this research paper, we applied fixed-point technique with cubic B-Spline scaling function to obtain a numerical solution for a set of non-linear integral equations without the need for algebraic systems. Using the numerical examples and the obtained results, as well as equation number (5), we found that the results are highly accurate and closely approximate the exact solution. We compared these results with those obtained from previous studies and observed the accuracy of this method. Since the accuracy of this method depends on using larger values for G and K, we note that the accuracy will improve with larger values. We continue the iterations until we reach a precise solution that approaches the true solution. This method is free from scenarios, problems, or excessive computational costs, and it is applicable to equations with larger and more complex dimensions. This method has no adverse effects on more complex equations.

8. References

[1] Arbabi F, Li FJJoSE. Buckling of variable cross-section columns: integral-equation approach. 1991;117(8):2426-41.

[2] Hochstadt H. Integral equations: John Wiley & Sons; 1989.

[3] Jerri AJ. Introduction to integral equations with applications: John Wiley & Sons; 1999.

[4] Kanwal RP. Linear integral equations: Springer Science & Business Media; 2013.

[5] Argyros IKJBotAMS. Quadratic equations and applications to Chandrasekhar's and related equations. 1985;32(2):275-92.

[6] Deimling K. Non-linear functional analysis: Springer Science & Business Media; 2013.

[7] Hu S, Khavanin M, Zhuang WJAA. Integral equations arising in the kinetic theory of gases. 1989;34(3-4):261-6.

[8] Precup R. Methods in non-linear integral equations: Springer Science & Business Media; 2002.

[9] Banaś J, Dhage BCJNAT, Methods, Applications. Global asymptotic stability of solutions of a functional integral equation. 2008;69(7):1945-52.

[10] Aghajani A, Jalilian YJCiNS, Simulation N. Existence and global attractivity of non-linear functional integral equation solutions. 2010;15(11):3306-12.

[11] Sommariva AJNA. A fast Nyström-Broyden solver by Chebyshev compression. 2005;38(1):47-60.

[12] Petryshyn WJJoM, Mechanics. Projection methods in non-linear numerical functional analysis. 1967;17(4):353-72.

[13] Pata V. Fixed point theorems and applications: Springer; 2019.

[14] Maleknejad K, Torabi P, Mollapourasl RJC, Applications Mw. Fixed point method for solving non-linear quadratic Volterra integral equations. 2011;62(6):2555-66.

[15] Maleknejad K, Nosrati Sahlan MJIJoCM. The method of moments for solution of second kind Fredholm integral equations based on B-spline wavelets. 2010;87(7):1602-16.

[16] Maleknejad K, Nedaiasl KJC, Applications Mw. Application of Sinc-collocation method for solving a class of non-linear Fredholm integral equations. 2011;62(8):3292-303.

[17] Maleknejad K, Karami M, Rabbani MJAm, computation. Using the Petrov–Galerkin elements to solve Hammerstein integral equations. 2006;172(2):831-45.

[18] Maleknejad K, Almasieh H, Roodaki MJCiNS, Simulation N. Triangular functions (TF) method for solving non-linear Volterra–Fredholm integral equations. 2010;15(11):3293-8.

[19] KtMAR S, SLOAN IJMC. A new collocation-type method for Hammerstein equations. 1987;48:585-93.

[20] Kelley CJTJoIE. Approximation of solutions of some quadratic integral equations in transport theory. 1982:221-37.

[21] Kaneko H, Noren RD, Padilla PAJJoC, Mathematics A. Superconvergence of the iterated collocation methods for Hammerstein equations. 1997;80(2):335-49.

[22] Guoqiang HJJoc, mathematics a. Extrapolation of a discrete collocation-type method of Hammerstein equations. 1995;61(1):73-86.

[23] Guoqiang H, Liqing ZJAm, computation. Asymptotic error expansion of a collocation-type method for Hammerstein equations. 1995;72(1):1-19.

[24] Brutman LJNA. Applying the generalized alternating polynomials to the numerical solution of Fredholm integral equations. 1993;5:437-42.

[25] Borzabadi AH, Fard OSJJoC, Mathematics A. A numerical scheme for a class of non-linear Fredholm integral equations of the second kind. 2009;232(2):449-54.

[26] Atkinson KE. The numerical solution of integral equations of the second kind: Cambridge University Press; 1997.

[27] Atkinson K, Han W. Theoretical numerical analysis: Springer; 2005.

[28] Pour-Mahmoud J, Rahimi-Ardabili MY, Shahmorad SJAM, Computation. Numerical solution of the system of Fredholm integro-differential equations by the Tau method. 2005;168(1):465-78.

[29] Maleknejad K, Arzhang AJAM, Computation. The numerical solution of the Fredholm singular integro-differential equation with Cauchy kernel using the Taylor-series expansion and Galerkin method. 2006;182(1):888-97.

[30] Dinghua XJJoSU. Numerical solutions for non-linear Fredholm integral equations of the second kind and their superconvergence. 1997;1(2):98-104.

[31] Babolian E, Shahsavaran AJJoC, Mathematics A. Numerical solution of non-linear Fredholm integral equations of the second kind using Haar wavelets. 2009;225(1):87-95.

[32] Maleknejad K, Karami MJAm, computation. Numerical solution of non-linear Fredholm integral equations by using multiwavelets in the Petrov–Galerkin method. 2005;168(1):102-10.

[33] Hackbusch W, Sauter SAJAom. On the efficient use of the Galerkin method to solve Fredholm integral equations. 1993;38(4):301-22.

[34] Xiao J-Y, Wen L-H, Zhang DJAM, Computation. Solving second kind Fredholm integral equations by periodic wavelet Galerkin method. 2006;175(1):508-18.

[35] Atkinson KEJSJoNA. The numerical evaluation of fixed points for completely continuous operators. 1973;10(5):799-807.

[36] Han GJBNM. Asymptotic error expansion for the Nyström method for non-linear Fredholm integral equations of the second kind. 1994;34(2):254-61.

[37] Babolian E, Abbasbandy S, Fattahzadeh FJAM, Computation. A numerical method for solving a class of functional and two-dimensional integral equations. 2008;198(1):35-43.

[38] Ebrahimi N, Rashidinia JJAM, Computation. A collocation method for linear and non-linear Fredholm and Volterra integral equations. 2015;270:156-64.

[39] Biazar J, Ghazvini H, Eslami MJC, Solitons, Fractals. He's homotopy perturbation method for systems of integro-differential equations. 2009;39(3):1253-8.

[40] Rashed MJAM, Computation. Numerical solution of functional differential, integral, and integro-differential equations. 2004;156(2):485-92.

[41] Rashed MJAM, Computation. Numerical solutions of functional integral equations. 2004;156(2):507-12.

[42] Banaś J, Rzepka BJAML. An application of a measure of noncompactness in the study of asymptotic stability. 2003;16(1):1-6.

[43] Burton T, Zhang BJAML. Fixed points and stability of an integral equation: nonuniqueness. 2004;17(7):839-46.

[44] Borzabadi AH, Kamyad AV, Mehne HHJAM, Computation. A different approach for solving the non-linear Fredholm integral equations of the second kind. 2006;173(2):724-35.

[45] Babolian E, Fattahzadeh F, Raboky EGJAM, Computation. A Chebyshev approximation for solving non-linear integral equations of Hammerstein type. 2007;189(1):641-6.

[46] [Razzaghi M, Ordokhani YJMPiE. Solution of non-linear Volterra-Hammerstein integral equations via rationalized Haar functions. 2001;7(2):205-19.

[47] Sepehrian B, Razzaghi MJMPie. Solution of non-linear Volterra-Hammerstein integral equations via single-term Walsh series method. 2005;2005(5):547-54.