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A Novel Approach in Number Theory for Representing Large Numbers: The Arrow-Free Notation

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Keywords: Number Theory, Knuth Up-Arrow Notation, Conway Chained Arrow Notation, Hyperoperation.

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Abstract:

This article introduces a new notation for expressing extremely large numbers, based on the hyperoperation concept in group theory. The method employs a finite sequence of positive integers separated by specific notational symbols, allowing for concise representation through an arrow-free notation: (a_n^b), where b represents the number of copies of a, and n denotes the arrow's number described by a general formula. This recursive definition aims to replace the Knuth up-arrow notation and Conway chained arrow notation, which require the insertion of arrows between or within numbers. The new approach simplifies these expressions, eliminating the need for such symbols and providing a straightforward and concise method for representing large numbers. The aim was to develop a more efficient method, arrow-free notation, reducing the complexity and steps necessary with previous notations.

Keywords: Number Theory, Knuth Up-Arrow Notation, Conway Chained Arrow Notation, Hyperoperation.

نهج جديد في نظرية الأرقام لتمثيل الأعداد الكبيرة: الترميز الخالي من الأسهم

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الخلاصة:

يتناول هذا المقال ترميزًا جديدًا يتعلق بمفهوم العمليات الفائقة في نظرية الزمر. يتعامل مع طرق التعبير عن بعض الأعداد الكبيرة جدًا. هو ببساطة تسلسل محدود من الأعداد الصحيحة الموجبة مفصولة بترميز خالٍ من الأسهم حيث يكون b هو عدد نسخ a ، و n هو عدد الأسهم، والذي يوصف بالصيغة العامة؛ (a_n^b) كما هو الحال مع معظم الترميزات التوافقية، فإن التعريف تكراري. في هذا المقال، ينتهي الترميز بأن يكون العدد الأيسر مرفوعًا إلى قوة عدد صحيح (عادة ما يكون ضخمًا). تم تصميم هذه الطريقة لتحل محل الطريقتين المستخدمتين سابقًا لتمثيل الأعداد الكبيرة، وهما ترميز السهم الصاعد لـ Knuth وترميز السلسلة المتسلسلة لـ Conway. يبسّط هذا النهج الجديد التعبير عن الأعداد الكبيرة دون الحاجة إلى كتابة الأسهم بين الأعداد أو داخلها، كما كان مطلوبًا في الطرق السابقة. كان هدف هذا البحث هو تحقيق هذه الميزة بهذه الطريقة المبتكرة، التي تبسط عدة خطوات كانت مطلوبة سابقًا من الطريقتين الأخريين، مما يوفر طريقة أكثر بساطة ووضوحًا لتمثيل نفس الحالات. الطريقة بسيطة وواضحة مع خطوات قليلة، مما يجعل من السهل تعلمها وتطبيقها لمعالجة مختلف حالات الأعداد الكبيرة والكبيرة جدًا.

الكلمات المفتاحية: نظرية الأعداد، ترميز السهم المتصاعد لـKnuth ، ترميز أسهم السلسلة لـConway ، العمليات العددية.

1. Introduction:

The notation x^y for tetration, introduced by Hans Maurer in 1901, has evolved significantly over the years [1]. In 1947, Reuben Louis Goodstein coined the term tetration and named other hyper-operations [2]. Andrzej Grzegorczyk further advanced this field in 1953, leading to the sequence of hyper-operations sometimes being referred to as the Grzegorczyk hierarchy [2,3]. Rudy Rucker's 1995 publication, *Infinity and the Mind*, popularized this notation extensively [4], despite his credit sometimes being mistakenly attributed to its first use. In 1976, Donald E. Knuth developed the Up-arrow notation for hyper-operations [4], which has found significant success on the internet due to its convenience in representing tetration as "x^^y". More recently, in 1996, John Horton Conway and Richard K. Guy introduced another hyper-operation notation in their book *The Book of Numbers* [2,3]. Large numbers, which significantly exceed those typically encountered in everyday life, such as in simple counting or monetary transactions, are frequently found in fields like mathematics, cosmology, cryptography, and statistical mechanics. In 2002, Jonathan Bowers developed Array Notation, which extends beyond hyperoperations while still encompassing them [3]. Bowers' notation includes an infix notation known as extended operator notation, equivalent in all respects to the current de facto standard notation for hyperoperations [4,5].

In mathematics, large numbers, defined as numbers exceeding one million, are typically represented either using exponents, such as (10^9, or 10^9), or by terms like billion or thousand million, which vary across different numeration systems. The American system of numeration for denominations above one million was originally based on a French system. However, in 1948, the French system was modified to align with the German and British systems [6,7]. In the American system, each denomination above one thousand million (the American billion) is 1,000 times the preceding one (for example, one trillion equals 1,000 billion; one quadrillion equals 1,000 trillions). Conversely, in the British system, each denomination is 1,000,000 times the preceding one (for example, one trillion equals 1,000,000 billion), with the exception of the term "milliard," which is sometimes used for one thousand million. In recent years, British usage has increasingly reflected the widespread adoption of the American system [8].

Scientific notation was devised to handle the vast range of values that arise in scientific research. For example, 1.0×10^9 denotes one billion, or a 1 followed by nine zeros: 1,000,000,000. Its reciprocal, 1.0×10^{-9}, represents one billionth, or 0.000000001. Using 10^9 instead of nine zeros alleviates the reader's effort and minimizes the risk of errors associated with counting a long series of zeros to determine the magnitude of the number [9]. In addition to scientific (powers of 10) notation, the following examples illustrate the systematic nomenclature of large numbers based on the short scale. The Avogadro constant [10], for example, is the number of "elementary entities" (usually atoms or molecules) in one mole; the number of atoms in 12 grams of carbon-12 is approximately 6.022×10^{23}, or 602.2 sextillion. The lower bound on the game-tree complexity of chess, also known as the "Shannon number" (estimated at around 10^{120}), or 1 novemtrigintillion. Rayo's number, named after Agustín Rayo, is a large number that has been claimed to be the largest named number [11]. At MIT on January 26, it was first defined during a "big number duel", 2007[12]. Graham's number, which surpasses what can be represented using power towers (tetration), can nonetheless be expressed using layers of Knuth's up-arrow notation. Nevertheless, there are very specific methods used to write out such huge numbers. Gödel numbers, as well as similar huge numbers used to represent bit-strings in algorithmic information theory, are extraordinarily large, even for mathematical statements of moderate length. One of the earliest mentions of hyperoperations was by Albert Bennett in 1914, where he developed a portion of his theory on commutative hyperoperations [11,12]. Twelve years later, Wilhelm Ackermann defined a function, denoted as (ϕ), which resembled a sequence of hyperoperations. In a 1947 paper, the Greek mathematician Ruben Goodstein introduced a sequence of operations now known as hyperoperations [12]. He extended these operations beyond exponentiation by adding the subsequent steps, which he suggested be called tetration and pentation. As a function with three arguments, this is recognized as a variant of the original Ackermann function. Nevertheless, some pathological numbers surpass the magnitude of Gödel numbers associated with typical mathematical propositions [10,11]. In general, there are very specific methods used to write out such extremely large numbers, such as:

- Scientific Notation: A way of expressing numbers as a product of a coefficient and a power of 10[11]. For example, 3.2×10^{15} represents 3,200,000,000,000,000.
- Knuth's Up-Arrow Notation: A technique to represent exemplifying very large integers using arrows to denote iterated exponentiation [11]. For instance, $3 \uparrow\uparrow 3$ is $\left(3^{3^3}\right)$, or (3^{27}), which is (7,625,597,484,987).
- Conway's Chain Arrow Notation: An extension of up-arrow notation, useful for representing even larger numbers [12]. For example, $3 \rightarrow 3 \rightarrow 3$ represents a very large number.
- Steinhaus-Moser Notation: A way to represent very large numbers using polygons and the Ackermann function [13]. For example, 2 inside a triangle ($\Delta 2$) is equivalent to 2^2 or 4, but the notation can escalate quickly to represent vastly larger numbers.
- Hyper-E notation: A notation for expressing extremely large numbers, using "E" to denote a level of exponentiation beyond standard operations [14]. For example, $3E3$ represents (3^{27}).
- Tree notation: Used in certain contexts like Kruskal's tree theorem [11,15], where numbers grow rapidly with tree structures and graph-related operations.

A standardized method for writing very large numbers facilitates their easy sorting in ascending order and provides a clear understanding of how much larger one number is compared to another. These notations represent functions of integer variables that grow at an exceptionally rapid rate as the integer values increase. By applying these functions recursively with large integer arguments, it is possible to construct functions that grow even more quickly. However, functions characterized by vertical asymptotes are not appropriate for defining very large numbers, despite their rapid growth. This is due to the fact that such functions necessitate arguments approaching the asymptote, which involves working with extremely small values, such as reciprocals, rather than directly addressing the construction of large numbers.

2. Materials and methods:

In this paper, we introduce Al-Ossmi's notation, a novel method for expressing extremely large numbers, named after its creator, Al-Ossmi. This notation aims to compactly represent large numbers by providing a clear structure that indicates the base, the level of iteration, and the depth of the operation. By doing so, it offers an efficient and unambiguous method for handling vast numerical values, making it a valuable tool for mathematicians and computer scientists.

Al-Ossmi's arrow-free notation is defined as; $a_n^{(\boldsymbol{b}\ copies\ of\ \boldsymbol{a})}$, where:

- a: The base number.
- b: The number of iterations or the height of the power tower.
- n: The level of operation or the number of arrows in Knuth's notation.

Also, the arrow-free notation developed to be extended to deal with the pentation (iterated tetration), including more complex structures, representing additional levels of nested operations where d and c are variables, such as;

$$a_n^{a_n^{(a_n^a)}}\ ,\quad a_n^{\left(a_n^{\left(a_n^{(a_n^a)}\right)}\right)}\ , \text{ or }\ a_n^{b.c.d.e.f.g}\ ,$$

Al-Ossmi's arrow-free notation simplifies the representation of very large numbers by using a compact form that corresponds to $(a \uparrow^n b)$ as in Knuth's up-arrow notation **[15]**, and in Conway's chained arrow notation $a \to b \to n$ **[11,16]**.

3. Results:

The method of writing and representing extremely large numbers is essential in mathematics, as it helps simplify and understand complex calculations and allows for the precise and concise expression of enormous quantities. Using specific symbols and notations to represent these numbers facilitates their handling in various mathematical fields, enhancing the efficiency of computations and analyses. These methods provide valuable tools for mathematicians and scientists who deal with numbers that exceed traditional representation capabilities. In our research, we will focus on the two most well-known methods, Knuth's up-arrow notation and Conway's chained arrow notation.

3.1 Knuth's up-arrow notation: This notation, introduced by Donald Knuth in 1976, is a system designed for representing very large integers **[15,16]**. It also illustrates the representation of numbers and the execution of arithmetic operations within this base system.

***Definition*:** For all integers a, b, n with $a \geq 0, n \geq 1, b \geq 0$, [16], the up-arrow operators can be formally defined by:

$$a \uparrow^n b = \begin{cases} a^b, & if\ n = 1; \\ 1, & if\ n > 1\ and\ b = 0; \\ a \uparrow^{(n-1)} b\big(a \uparrow^n (b-1)\big), & otherwise \end{cases}$$

This definition employs exponentiation as the base case and tetration as repeated exponentiation, aligning with the hyperoperation sequence but excluding the fundamental operations of succession, addition, and multiplication. The sequence begins with a unary operation (the successor function), for ($n = 0$) and progresses through binary operations such as addition ($n = 1$), multiplication ($n = 2$), exponentiation ($n = 3$), as well as tetration ($n = 4$), and pentation ($n = 5$), among others. This framework is often used to represent hyperoperations with arrows, for example;

- The single arrow (↑) represents exponentiation (iterated multiplication);
$$3 \uparrow 2 = 3 \times 3 = 3^2 = 9$$
- The double arrow (↑↑) represents tetration (iterated exponentiation);
$$3 \uparrow\uparrow 4 = 3 \uparrow 3 \uparrow (3 \uparrow 3) = 3 \uparrow 3 \uparrow (3^3) = 3 \uparrow 3 \uparrow (27)$$
- The triple arrow (↑↑↑) represents pentation (iterated tetration);
$$3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3) = 3 \uparrow\uparrow 3 \uparrow\uparrow \left(3^{3^3}\right)$$

The up-arrow notation general definition is as follows (for $a \geq 0, n \geq 1, b \geq 0$):

$$a \uparrow^n b = a[n+1]b$$

Here, ↑ stands for *n* arrows, so for example: $3 \uparrow\uparrow\uparrow\uparrow 5 = 3 \uparrow^4 5$.

Square brackets are another notation for hyperoperations. Exponentiation for a natural power b is defined as iterated multiplication, which Knuth denoted by a single up-arrow:

$a \uparrow b = a^b$, for example, $3 \uparrow 4 = 3 \times 3 \times 3 \times 3 = 3^4 = 81$

Tetration is well-defined as iterated exponentiation, which Knuth denoted by a "double arrow":

$a \uparrow\uparrow b = a \uparrow \big(a \uparrow (\ldots \uparrow a)\big)$, b copies of a's.

Example; $3 \uparrow\uparrow 4 = 3 \uparrow 3 \uparrow (3 \uparrow 3) = 3^{\left(3^{3^3}\right)} = 3^{(7,625,597,484,987)}$

Expressions are evaluated from right to left because the operators are right-associative. This results in very large numbers, but the hyperoperator sequence continues beyond pentation. Pentation, defined as iterated tetration, is represented by the "triple arrow":

$a \uparrow\uparrow\uparrow b = a \uparrow\uparrow \big(a \uparrow\uparrow (\ldots \uparrow\uparrow a)\big)$, hence b copies of a's.

Example, $3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3)$

$$= 3 \uparrow\uparrow 3 \uparrow\uparrow \left(3^{3^3}\right) = 3 \uparrow\uparrow 3 \uparrow\uparrow (7,625,597,484,987)$$

Hexation, which is the iteration of pentation, is denoted using the "quadruple arrow" notation: $a \uparrow\uparrow\uparrow\uparrow b = a \uparrow\uparrow\uparrow \big(a \uparrow\uparrow\uparrow (\ldots \uparrow\uparrow\uparrow a)\big)$, b copies of a's.

Example, $3 \uparrow\uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3) = 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (3 \uparrow\uparrow 3)\big)$,

$$= 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3)\big)$$

$$= 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (7{,}625{,}597{,}484{,}987)\big) \quad \text{and so on.}$$

The general principle is that an n-arrow operator unfolds into a right-associative sequence of $(n-1)$-arrow operators. For instance, expressing $a \uparrow\uparrow b$ in traditional superscript notation results in a power tower. If b is a variable or is exceptionally large, the power tower may be denoted with ellipses and an annotation specifying the height of the tower.

$$\underset{n}{a \uparrow\uparrow \cdots \uparrow b} = \underbrace{a \uparrow \cdots \uparrow \big(a \uparrow \cdots \uparrow (\ldots \uparrow \cdots \uparrow a)\big)}_{b \text{ copies of } a}$$

Examples:

$$\mathbf{3} \uparrow\uparrow\uparrow \mathbf{2} = 3 \uparrow\uparrow 3 = 3 \uparrow 3 \uparrow 3 = 3^{3^3} = 7{,}625{,}597{,}484{,}987$$

$$\mathbf{3} \uparrow\uparrow\uparrow \mathbf{3} = 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3), with;\ 3 \uparrow 3 \uparrow 3\ copies\ of\ 3.$$

Where; $3 \uparrow 3 \uparrow 3 = 3^{3^3} = 7{,}625{,}597{,}484{,}987\ copies\ of\ 3.$

$$\mathbf{3} \uparrow\uparrow\uparrow\uparrow \mathbf{4} = 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3) = 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (3 \uparrow\uparrow 3)\big)$$

$$= 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3)\big)$$

$$= 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \Big(3 \uparrow\uparrow \big(3^{3^3}\big)\Big)$$

$$= 3 \uparrow\uparrow\uparrow 3 \uparrow\uparrow\uparrow \big(3 \uparrow\uparrow (7{,}625{,}597{,}484{,}987\ )\big)$$

where; $3 \uparrow 3 \uparrow 3 = 3^{3^3} = 7{,}625{,}597{,}484{,}987\ copies\ of\ 3.$

In this notation system, the expression $a \uparrow\uparrow b$ can be represented as a stack of power towers, where each level of the stack illustrates the magnitude of the level above it. If b is a variable or excessively large, this stack may be denoted using ellipses with an annotation indicating its height. Similarly, $a \uparrow\uparrow\uparrow b$ can be represented by multiple columns of these power tower stacks, where each column represents the number of power towers in the stack to its left. This up-arrow notation simplifies the representation of these diagrams while maintaining a geometric framework, such as tetration towers. Although this notation can handle very large numbers, the hyperoperator sequence extends beyond this scope. For extremely large numbers, Knuth's multiple arrows become impractical. In such cases, the n-arrow operator ($\uparrow^n$) is useful for describing sequences with a variable number of arrows, or hyperoperators. For numbers that surpass even this notation's capabilities, Conway's chained arrow notation can be employed. While a chain of three elements is comparable to other notations, a chain of four or more elements significantly enhances its capacity.

3.2 Conway chained-arrow notation: Similar to Knuth's, the chained-arrow notation has several properties that are similar to exponentiation, as well as properties that are specific to the operation and are gained from exponentiation. Conway's chained-arrow notation, invented by mathematician John Horton Conway, is a method for representing extraordinarily large numbers **[12]**. This notation involves a finite sequence of positive integers separated by rightward arrows, such as:

$$2 \rightarrow (3 \rightarrow 4 \rightarrow 5) \rightarrow 6 .$$

Like many combinatorial notations, Conway chained arrow notation is defined recursively. Eventually, the notation simplifies to the leftmost number being raised to a very large integer power.

***Definition1*:** A "Conway notation" is defined as follows:

- Any positive integer can be represented as a chain of length 1.
- A chain of length n, followed by a right-arrow → and a positive integer, together form a chain of length $(n+1)$.

Any chain represents an integer, according to the six rules below **[10]**. Two chains are said to be equivalent if they represent the same integer. Let a, b, n denote positive. Then:

1. An empty chain (or a chain of length 0) is equal to 1.
2. The chain a represents the number a.
3. The chain $(a \rightarrow b)$ represents the number (a^b).
4. The chain $(a \rightarrow b \rightarrow n)$ represents the number $a \uparrow^n b$.

$$a \uparrow\uparrow b = a \rightarrow b \rightarrow n$$

Examples can become quite intricate rapidly. Here are a few simple instances:

$$2^{2^2} = 2 \uparrow\uparrow 3 = 2 \rightarrow 3 \rightarrow 2$$

$$4 \uparrow\uparrow 3 = 4 \rightarrow 3 \rightarrow 2$$

$$5 \uparrow\uparrow\uparrow 2 = 5 \rightarrow 2 \rightarrow 3$$

Arrow chains do not represent the iterative application of a binary operator. Instead, chains of other infixed symbols $(a \uparrow^n b)$ can frequently be considered in fragments $(a \rightarrow b \rightarrow n\,))$ without a change, for example:

$$2^{3^2} = 2^9 = 2 \rightarrow (3 \rightarrow 2)$$
$$(2^3)^2 = 8^2 = (2 \rightarrow 3) \rightarrow 2$$

$$2 \uparrow\uparrow\uparrow 4 = 2 \uparrow\uparrow 2 \uparrow\uparrow (2 \uparrow\uparrow 2) = 2 \uparrow\uparrow 2 \uparrow\uparrow (2 \uparrow 2) = \boldsymbol{a \rightarrow b \rightarrow n = 2 \rightarrow 4 \rightarrow 3}$$

The sixth rule in Conway's chained-arrow notation is essential. It dictates that for a sequence with four or more elements, ending in a number 2 or higher, the sequence is transformed into one of the same lengths but with a significantly larger penultimate element **[12]**. The last element of the sequence is reduced, which simplifies the sequence according to Knuth's detailed procedure. This reduction process continues until the sequence is condensed to three elements, where the fourth rule completes the recursion.

4. The Arrow-Free Notation

The need for new arithmetic operations on very large numbers arises from various practical and theoretical considerations in fields such as science, technology, and mathematics. In this paper, we introduce a modern approach with an arrow-free notation that allows for the representation of numbers so large they are beyond common human experience. Al-Ossmi's arrow-free notation, named after its inventor, represents a function of integer variables that escalate at an exceptionally rapid rate as these integers increase. This notation allows for the recursive construction of increasingly faster-growing functions by applying it with large integer arguments.

***Definition 2*:** For all non-negative integers a, b, n with $a \geq 0, n \geq 1, b \geq 0,$ the Al-Ossmi's arrow-free operators can be formally defined by:

$$\boldsymbol{a_n^b} = \begin{cases} a^b\,, & if\ n = 1 \\ 1, & if\ n > 1\ and\ b = 0; \\ a, & if\ n > 1\ and\ b = 1; \\ a_{(n-1)}\ (a_n(b-1)), & otherwise \end{cases}$$

It is important to note that Knuth did not define the "nil-arrow" operator $a \uparrow^0$, whereas Al-Ossmi's notation does include this concept. Furthermore, Al-Ossmi's notation can be extended to negative indices $(n \geq -2)$ to align with the entire hyperoperation sequence, albeit with a delay in the indexing, which can be formed as:

$$\boldsymbol{a} \uparrow^{(n-1)} b = \boldsymbol{a}_{(n-1)}^{b}\ ,\quad for\ (n \geq 0)$$

For $(n = 1)$,we obtain the ordinary exponentiation, hence this definition uses exponentiation; $\boldsymbol{a_1^b} = \boldsymbol{a^b}$,as the base case, and tetration; $\boldsymbol{a_2^b}$ as repeated exponentiation. This approach aligns with the hyperoperation sequence but excludes the three fundamental operations of succession, addition, and multiplication. Alternatively, one may choose to define multiplication as follows:

$$a \uparrow^1 1 = a^1 = \boldsymbol{a_1^1} = a$$

$$a \uparrow^1 b = a^b = \boldsymbol{a_1^b} = a^b$$

$$a \uparrow^0 b = a \times b = \boldsymbol{a_0^b} = a.b$$

The up-arrow operation is right-associative, meaning that is, $a \uparrow\ b \uparrow c$, is understood to be, $a \uparrow\ (b \uparrow c)$, instead of, $(a \uparrow\ b) \uparrow c$, while it is donated by Al-Ossmi's arrow-free notation as:

$$a^{\left(b^{\left(c^{d}\right)}\right)} = a \uparrow (b \uparrow c \uparrow d) = a \uparrow (b.c.d) = \boldsymbol{a}^{(\boldsymbol{b.c.d})}$$

$$a \uparrow b \uparrow c = a \uparrow (b \uparrow c) = \boldsymbol{a}_1^{(b\uparrow c)} = \boldsymbol{a}_1^{(b.c)} = \boldsymbol{a}^{(b.c)}$$

$$a \uparrow\uparrow b \uparrow c = a \uparrow\uparrow (b \uparrow c) = \boldsymbol{a}_{\boldsymbol{2}}^{(b\uparrow c)} = \boldsymbol{a}_2^{(b.c)}$$

$$a \uparrow\uparrow\uparrow b \uparrow c = a \uparrow\uparrow\uparrow (b \uparrow c) = \boldsymbol{a}_{\boldsymbol{3}}^{(b\uparrow c)} = \boldsymbol{a}_3^{(b.c)}$$

Then exponentiation becomes repeated multiplication, $a \uparrow b \uparrow c \uparrow d$ in form of $(a.b.c.d)$, which means; $a^{\left(b^{\left(c^{d}\right)}\right)}$. The formal definition would be donated by Al-Ossmi's arrow-free notation as iterated exponentiation of a power tower of *b*:

$$a \uparrow b \uparrow c \uparrow d = a \uparrow (b \uparrow c \uparrow d) = \boldsymbol{a}_1^{(b\uparrow c\uparrow d)} = \boldsymbol{a}_1^{(b.c.d)} = \boldsymbol{a}^{(\boldsymbol{b.c.d})},$$

$$a \uparrow b \uparrow c \uparrow d \uparrow e = a \uparrow (b \uparrow c \uparrow d \uparrow e) = \boldsymbol{a}_1^{(b\uparrow c\uparrow d\uparrow e)} = \boldsymbol{a}_1^{(b.c.d.e)} = \boldsymbol{a}^{(\boldsymbol{b.c.d.e})},$$

$$a \uparrow\uparrow b \uparrow c \uparrow d \uparrow e = a \uparrow\uparrow (b \uparrow c \uparrow d \uparrow e) = \boldsymbol{a}_{\boldsymbol{2}}^{(b\uparrow c\uparrow d\uparrow e)} = \boldsymbol{a}_{\boldsymbol{2}}^{(b.c.d.e)},$$

Hyperoperations extend arithmetic operations beyond exponentiation. Specifically, exponentiation, defined as iterated multiplication for a natural power b, is denoted by a single up-arrow in Knuth's notation. In Al-Ossmi's arrow-free notation, this operation is represented when $(n = 1)$, as:(a_1^b) which is also donated by: a^b. In this new notation, expressions are evaluated from right to left due to the right-associative nature of the operators. Tetration, which is defined as iterated exponentiation, is represented in Knuth's notation using a "double arrow":

$$a \uparrow\uparrow b = a_2^b = \boldsymbol{a}_{\boldsymbol{n}-1}^{(\boldsymbol{b})} = \boldsymbol{a}^{(\boldsymbol{b}\ copies\ of\ \boldsymbol{a})},$$

For example;

$$\mathbf{3} \uparrow\uparrow \mathbf{4} = 3 \uparrow 3 \uparrow 3 \uparrow 3 = \mathbf{3}^{\mathbf{3}^{\mathbf{3}^{\mathbf{3}}}}$$

$$3 \uparrow\uparrow 4 = \mathbf{3}_{\mathbf{2}}^{\mathbf{4}} = \mathbf{3_1 3_1 3_1 3_1} = \mathbf{3}^{\mathbf{3}^{\mathbf{3}^{\mathbf{3}}}}$$

According to this definition, examples of numbers which are written by Knuth's can be rewritten out by the Al-Ossmi's arrow-free notation as following:

$$3 \uparrow\uparrow \mathbf{2} = 3 \uparrow 3 = 3_2^{\mathbf{2}} = 3_1 3_1 = \mathbf{3}^{\mathbf{3}}$$

$$3 \uparrow\uparrow \mathbf{3} = 3 \uparrow 3 \uparrow 3 = 3_2^{\mathbf{3}} = 3_1 3_1 3_1 = \mathbf{3}^{\mathbf{3}^{\mathbf{3}}}$$

$$3 \uparrow\uparrow \mathbf{4} = 3 \uparrow 3 \uparrow 3 \uparrow 3 = 3_2^{\mathbf{4}} = 3_1 3_1 3_1 3_1 = \mathbf{3}^{\mathbf{3}^{\mathbf{3}^{\mathbf{3}}}}$$

$$3 \uparrow\uparrow \mathbf{5} = 3 \uparrow 3 \uparrow 3 \uparrow 3 \uparrow 3 = 3_2^{\mathbf{5}} = 3_1 3_1 3_1 3_1 3_1 = \mathbf{3}^{\mathbf{3}^{\mathbf{3}^{\mathbf{3}^{\mathbf{3}}}}}$$

This process results in exceedingly large numbers, yet the hyperoperator sequence extends further. Pentation, which is defined as iterated tetration and denoted by Knuth using the "triple arrow,"$a \uparrow\uparrow\uparrow b$, and by the arrow-free notation, it is simply represented as: a_3^{b}. According to this definition:

$$3 \uparrow\uparrow\uparrow \mathbf{2} = 3_3^{\mathbf{2}}$$

$$10 \uparrow\uparrow\uparrow\uparrow 5 = 10_4^{\mathbf{5}}$$

Hexation and beyond, which are defined as iterated pentation, are represented using Knuth's "quadruple arrow" notation as $a \uparrow\uparrow\uparrow\uparrow b$. In Al-Ossmi's notation, these operations are expressed as a_4^b. According to this definition:

$$3 \uparrow\uparrow\uparrow\uparrow 2 = 3_4^{\mathbf{2}}$$

$$10 \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow 5 = 10_6^{\mathbf{5}}$$

The sequence of operations begins with a unary operation (hence the successor function for $(n = 0)$ and extends through binary operations such as addition $(n = 1)$, multiplication $(n = 2)$, exponentiation $(n = 3)$, tetration $(n = 4)$, pentation $(n = 5)$, and so on. Different notations have been used to represent these hyperoperations. In Al-Ossmi's notation, the base is represented by a, with b indicating the number of copies of a. Therefore, an n-arrow operator in Al-Ossmi's notation translates into a right-associative series of n-arrow operators.
Symbolically:

$$a \underbrace{\uparrow\uparrow \cdots \uparrow}_{b \text{ copies of } a} b = \boldsymbol{a_n^b},$$

Examples:

$$2 \uparrow\uparrow\uparrow 2 = 2 \uparrow\uparrow \mathbf{2} = 2_3^{\mathbf{2}} = 2_2^{\mathbf{2}}$$

$$3 \uparrow\uparrow\uparrow 2 = 3 \uparrow\uparrow \mathbf{3} = 3_3^{\mathbf{2}} = 3_2^{\mathbf{3}}$$

$$3 \uparrow\uparrow\uparrow \mathbf{3} = 3_3^{\mathbf{3}}$$

$$3 \uparrow \mathbf{6} = 3 \uparrow^{\mathbf{23}} \mathbf{6} = 3_{\mathbf{23}}^{\mathbf{6}}$$

$$3 \uparrow^{10^{100}} \mathbf{4} = 3_{10^{(100)}}^{\mathbf{4}} = 3_{10.100}^{\mathbf{4}}$$

$$3 \uparrow^{10^{23.5}} \mathbf{4} = 3_{10^{(23.5)}}^{\mathbf{4}} = 3_{10.(23.5)}^{\mathbf{4}}$$

$$3 \uparrow^{10^{23^5}} \mathbf{4} = 3_{10^{(23^5)}}^{\mathbf{4}} = 3_{10.23.5}^{\mathbf{4}}$$

This flexibility of Al-Ossmi's notation with forms; $\boldsymbol{a_n^b}$, $\boldsymbol{a_n^{b.c}}$ or $\boldsymbol{a_n^{b.c.d.e.f.g.h}}$ **,** all allow for a compact and structured way to represent extremely large numbers. These power tower $(b.c.d.e)$ are components to describe complex and extended depth and extend the depth further, representing additional layers of power towers or nested operations. By accurately reflecting the base, height (with nuanced depth like $b.c.d$, indicating an extended height in the power tower), and level of exponentiation, it simplifies understanding and interpreting large numbers. The use of multiple components separated by dots allows for a detailed and nuanced representation of the structure of the large number. This notation captures the nuances of the structure of these numbers, including the base, height, depth, and level of operations, making it easier to understand and work with extremely large values.

4.1 Compared with Conway's chained notation: Let $(a > 1)$ is the base, $(b > 1)$ is the tower power, and $(n \geq 1)$ is the arrow's number, then Al-Ossmi's arrow-free notation can be written as:

$$a \uparrow^n b = a \to b \to n = \boldsymbol{a_n^b} = (\boldsymbol{a_n} \dots \boldsymbol{b}\ copies\ of\ \boldsymbol{a}) = \boldsymbol{a}^{(\boldsymbol{b}\ copies\ of\ \boldsymbol{a})}$$

When the number is power by n times, then the Al-Ossmi's notation presents the power tower by adding the base a, arrows number $= n$, and $b =$ number of tower of a , for example:

$$a \to a \to 1 = \boldsymbol{a_1^a} = \boldsymbol{a^a}$$

$$2 \to 2 \to 1 = \boldsymbol{2_1^2} = \boldsymbol{2^2} = \boldsymbol{4}$$

where $\boldsymbol{2_1^2}$ represents 2 with one level of iteration (simple exponentiation), and a height of 2.

$$2 \to (2 \to 2 \to 1) \to 2 = \boldsymbol{2_2^3} = \boldsymbol{2_1 2_1 2_1} = \boldsymbol{2^{2^2}} = \boldsymbol{2^4} = \boldsymbol{16}$$

where $\boldsymbol{2_2^3}$ represents 2 with one level of iteration (simple exponentiation), and a height of 3.

$$2 \to (2 \to 2 \to 2) \to 1 = 2 \to (16) \to 2 = \boldsymbol{2_2^4} = \boldsymbol{2_1 2_1 2_1 2_1} = \boldsymbol{2^{2^{2^2}}} = \boldsymbol{65,536}$$

where $\boldsymbol{2_2^4}$ represents 2 with one level of iteration (simple exponentiation) and a height of 4.

and form rule of (3); we can write it out as; $a_2^{\mathrm{b}} = \boldsymbol{a}^{(\boldsymbol{b}\ copies\ of\ \boldsymbol{a})}$, which is applied in this example:

$$\boldsymbol{10^{10^{10^{11}}}} = 10 \to 10 \to 1 \to (10 \to 11 \to 1) \to 2 = \boldsymbol{10_2^{3^{(11)}}}$$

$$= \boldsymbol{10_1 10_1 10_1^{(11)}} = \boldsymbol{10_2^{3^{(11)}}} = \boldsymbol{10_2^{(3.11)}}$$

where $\boldsymbol{10_2^{3.11}}$ represents 10 with 2 levels of iteration (simple exponentiation), and a height of 10 to the power of 11 and raised 3 times.

$$\boldsymbol{3^{3^{(56)^2}}} = 3 \to (3 \to (56 \to 2 \to 1) \to 1) \to 1 = \boldsymbol{3_1^{3^{(56)^2}}}$$

$$= \boldsymbol{3_1 3_1^{(56)^2}} = \boldsymbol{3^{3^{(56)^2}}} = \boldsymbol{3_1^{3.(3136)}} = \boldsymbol{3_2^{2.(3136)}}$$

Example1:

$$\boldsymbol{10^{10^{10^{10^{10}}}}} = \boldsymbol{10_2^5} = 10_1 10_1 10_1 10_1 10_1$$

Example2:

$$\boldsymbol{10^{10^{303}}} = \boldsymbol{10_2^{2^{(303)}}} = 10_1 10_1^{(303)}$$

Example3:

$$\boldsymbol{10^{2^{303}}} = \boldsymbol{10_1^{2^{(303)}}} = 10_1^{2.303} = 10^{2.303}$$

The difference between $\boldsymbol{10_1 10_1^{(303)}}$, regarding $\boldsymbol{10_1^{2.303}}$, is in how exponentially larger the exponent of $\boldsymbol{a_1^{b^{(c)}}}$ compared to the exponent of $\boldsymbol{a^{b^{(c)}}}$. The difference between $\boldsymbol{10_2^{2^{(303)}}}$ and

$\mathbf{10_1^{2.(303)}}$ lies in the magnitude of the exponent. The exponent itself, 10^{303}, is already a number with 304 digits (a 1 followed by 303 zeros). When you raise 10 to this power, you get a number with 10^{303} digits.

Al-Ossmi's free arrows notation a_n^b for expressing large numbers, $\boldsymbol{a}$ is the base number, $\boldsymbol{b}$ is the height or number of iterations in the power tower, and *n* is the level or number of arrows in Knuth's up-arrow notation.

Base Case ($n = \mathbf{1}$): Exponentiation: $a_n^b = a_1^{\boldsymbol{b}} = \boldsymbol{a^b}$

$$\text{Example: } 2 \uparrow 3 = \mathbf{2_1^3} = 2 \times 2 \times 2 = \mathbf{2^3} = 8$$

Two Levels ($n = \mathbf{2}$): Tetration (iterated exponentiation): $\boldsymbol{a_2^b} = a \uparrow\uparrow b$

$$\text{Example: } \mathbf{2 \uparrow\uparrow 3} = 2 \uparrow (2 \uparrow 2) = 2 \uparrow (2^2) = \mathbf{2}^{(\mathbf{2^2})}$$

From the rules of (2 & 3), we find; $\mathbf{2 \uparrow\uparrow 3} = \mathbf{2_2^3} = \mathbf{2_1(2_1 2_1)} = \mathbf{2}^{(\mathbf{2^2})} = 16$

For $a = 10$, $b = 3$ (height of the tower), and, $(n = 2)$, (level of operation), then form the rules of (2 and 3), we find;

$$10 \uparrow\uparrow 3 = \mathbf{10_2^3} = 10_1(10_1 10_1) = 10^{(10^{10})}$$

Three Levels ($n = \mathbf{3}$): Pentation (iterated tetration):

$$\boldsymbol{a_3^b} = a \uparrow\uparrow\uparrow b = a \uparrow\uparrow (\dots a \uparrow\uparrow a) \text{ , (with } b \text{ copies of } a\text{), then:}$$

$$\boldsymbol{a \uparrow\uparrow\uparrow b = a_n^b = a_{(n-1)}^b \left(a_{(n-1)}^{(b\ copies\ of\ a)}\right)},$$

Example, according to Knuth's;

$$\mathbf{2 \uparrow\uparrow\uparrow 3} = 2 \uparrow\uparrow (\mathbf{2 \uparrow\uparrow 2}) = 2 \uparrow\uparrow (\mathbf{2 \uparrow 2}) = 2 \uparrow\uparrow (\mathbf{2^2}) \equiv \mathbf{2 \uparrow\uparrow 4}$$

Then according to the Al-Ossmi's arrow-free notation's;

$$\mathbf{2 \uparrow\uparrow\uparrow 3 = 2_3^3},$$

$$\mathbf{2 \uparrow\uparrow (2^2) = 2 \uparrow\uparrow 4 = 2_2^{(2_1 2_1)} = 2_2^{(2^2)} = 2_2^{(4)} = 2_2^4}$$

$$\mathbf{2 \uparrow\uparrow\uparrow 3 \equiv 2 \uparrow\uparrow 4 \equiv 2_3^3 \equiv 2_2^4}$$

Let us prove that; $\boldsymbol{a} \uparrow\uparrow b = \boldsymbol{a} \uparrow (\boldsymbol{a} \uparrow \boldsymbol{a}) = \boldsymbol{a}^{(a^a)}$, which can be written as $\boldsymbol{a_2^b}$.

$$Example;\ \mathbf{10_2^3} = 10 \uparrow\uparrow 3 = 10 \uparrow (\mathbf{10 \uparrow 10}) = 10 \uparrow (\mathbf{10^{10}}) \equiv 10^{(\mathbf{10^{10}})}$$

$$\mathbf{10_2^3 = 10_1(10_1\ 10_1) = 10_1(10^{10}) = 10^{(10^{10})} = 10 \uparrow\uparrow 3}$$

By Al-Ossmi's arrow-free notation definition, represents $a \uparrow^n b\ as;\ a_n^b$, which matches our calculation above. Al-Ossmi's notation provides a concise and efficient way to represent very large numbers, combining the simplicity of Knuth's up-arrow notation with a clear structure for understanding the depth and height of power towers.

Example 4:

$$\mathbf{3\uparrow\uparrow\uparrow 4}$$

According to Knuth's;

$$\mathbf{3\uparrow\uparrow\uparrow 4} = 3\uparrow\uparrow 3\uparrow\uparrow(\mathbf{3\uparrow\uparrow 3}) = 3\uparrow\uparrow 3\uparrow\uparrow(\mathbf{3\uparrow 3\uparrow 3}) = 3\uparrow\uparrow 3\uparrow\uparrow\left(\mathbf{3^{3^{3}}}\right)$$

Then according to the Al-Ossmi's arrow-free notation's;

$$\mathbf{3\uparrow\uparrow\uparrow 4} = \mathbf{3_3^4} = \mathbf{3_2^3 3_2^{(3_1 3_1 3_1)}} = \mathbf{3_2^3 3_2^{\left(3^{3^{3}}\right)}} = \mathbf{3_2^3 3_2^{(7,625,597,484,987)}}$$

General Case $(n \geq 1)$: Iterated $(n-1)$ level operation:

$$a_n^b = a\uparrow^n b = a\uparrow^{(n-1)}\left(a\uparrow^{(n-1)}(\ldots a\uparrow^{n-1} a)\right), (b \text{ copies of } a)$$

Example 5;

$$\mathbf{5\uparrow\uparrow\uparrow 4} = 5\uparrow\uparrow(5\uparrow\uparrow(\mathbf{5\uparrow\uparrow 5})) = 5\uparrow\uparrow(5\uparrow\uparrow(\mathbf{5\uparrow 5\uparrow 5\uparrow 5\uparrow 5})),$$

$$\mathbf{5\uparrow\uparrow\uparrow 4} = \mathbf{5_3^4} = \mathbf{5_2^5}\left(5_2^{(5_1 5_1 5_1 5_1 5_1)}\right) = \mathbf{5_2^5}\left(5_2^{\left(5^{5^{5^{5^{5}}}}\right)}\right)$$

By defining the base (a), height (b), and level (n), it simplifies the notation and makes it easier to interpret and calculate extremely large values. Additional examples such as:

$$\mathbf{2}^{\left(2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}}}}}}}\right)} = 2\to 15\to 2\to 2\to 1 = \mathbf{2_2^{16}}$$

$$\mathbf{2}^{4^{4^{4^{4^{4}}}}} = 2\to(4\to 4\to 4\to 4\to 4)\to 1 = \mathbf{2_1^4}\left(\mathbf{4_2^5}\right)$$

$$\mathbf{2}^{4^{5^{5^{5}}}} = 2\to 4\to(5\to(5\to 5\to 1)\to 1)\to 1 = \mathbf{2_1^4}\left(\mathbf{5_2^3}\right)$$

$$\mathbf{2}^{3^{5^{5^{4}}}} = 2\to 3\to(5\to(5\to 4\to 1)\to 1)\to 1 = \mathbf{2_1^3}\left(\mathbf{5_2^2}\right)^{\mathbf{4}}$$

$$\boldsymbol{a}^{a^{a^{a^{(3.567)}}}} = a\to(a\to(a\to(a\to 1)\to 3.567\to 1)\to 1)\to 1$$

$$= \left(\boldsymbol{a_2^4}\right)^{(3.567)}$$

4.2 Discussion: For further clarification, a practical application of the new notation will be demonstrated on a large and diversed set of very big numbers, encompassing the widest possible range of number cases that can be written in this notation, as shown in the following **Table 1**.

Table 1: A set of detailed examples written by Al-Ossmi's arrow- free notation:

Al-Ossmi's notation	*Interpretation*
10_1^2	$10 \uparrow 2 = 10^2 = 100$
10_2^2	$10 \uparrow\uparrow 2 = 10^{10} = 10{,}000{,}000{,}000$
10_2^3	$10 \uparrow\uparrow 3 = 10 \uparrow 10 \uparrow 10$
10_2^4	$10 \uparrow\uparrow 4 = 10 \uparrow 10 \uparrow 10 \uparrow 10$
10_3^3	$10 \uparrow\uparrow\uparrow 3 = 10 \uparrow\uparrow 10 \uparrow\uparrow 10$
10_2^6	$10 \uparrow\uparrow 6 = 10 \rightarrow 6 \rightarrow 2$
10_3^{11}	$10 \uparrow\uparrow\uparrow 11$, This is simply 10 raised to the power of 11.
10_1^{64}	$10^{4^3} = 10 \uparrow (4 \uparrow 3) = 10 \uparrow 64 = 10 \rightarrow (64) \rightarrow 1$
3_2^3	$3 \uparrow\uparrow 3 = (3 \uparrow 3 \uparrow 3) = (7{,}625{,}597{,}484{,}987)$
$4_3^2 = 4_2^4$	$4 \uparrow\uparrow\uparrow 2 = 4 \uparrow\uparrow 4 = (4 \uparrow 4 \uparrow 4 \uparrow 4) = 4^{4^{4^4}}$
5_3^4	$5 \uparrow\uparrow\uparrow 4 = 5 \uparrow\uparrow 5 \uparrow\uparrow (5 \uparrow\uparrow 5)$, it is a tetration of 5 repeated 4 times.
3_4^3 $3_{G_n}^3$ $3_{G_{63}}^3$	Graham's Number G: $3 \uparrow^4 3 = 3 \rightarrow 3 \rightarrow 4$ $G_{n+1} = 3 \uparrow^{G_n} 3$ starting from G_1 *to* G_{64} $G_{n+1} = 3_3^{G_n}$, for 64 steps: $G_{63+1} = 3_3^{G_{63}}$
$3_{G_{64}}^{10}$	$3 \uparrow^{G_{64}} 10 = 3 \rightarrow 10 \rightarrow G_{64}$
5_{34}^4	$5 \uparrow^{34} 4 = 5 \rightarrow 4 \rightarrow 34$
$5_{34}^{\left(\mathbf{411^{300^{12}}}\right)}$	$5 \uparrow^{34} \left(411 \uparrow^{300^{12}}\right) = 5 \rightarrow (411 \rightarrow 300 \rightarrow 12) \rightarrow 34$
$10_2^{2^{\left(3^{4^{2^5}}\right)}}$	$10^{10^{\left(3^{4^{2^5}}\right)}} = 10 \uparrow 10 \uparrow (3 \uparrow 4 \uparrow 2 \uparrow 5)$
$10^{6.12.200.3}$	$10^{6^{12^{200^3}}} = 10 \uparrow (6 \uparrow 12 \uparrow 200 \uparrow 3)$
Al-Ossmi's notation	Interpretation
$10^{3.4.3402.5.3.4.2001}$	$8^{3^{4^{3402^{5^{3^{4^{2001}}}}}}} =$ $= 8 \uparrow \left(3 \uparrow \left(4 \uparrow (3402 \uparrow 5 \uparrow 3 \uparrow 4 \uparrow 2001)\right)\right)$
$10_2^{10.10.303}$	$10 \uparrow\uparrow 10 \uparrow 10 \uparrow 303 = 10 \uparrow\uparrow (10 \uparrow 10 \uparrow 303)$
$100_1^{10^3} = 100_1^{1000}$	$100^{1000} = 100^{10^3}$
100_2^5	$100^{100^{100^{100^{100}}}}$
$100_2^{4^{(12)}} = 100_2^{4.12}$	$100^{100^{100^{100^{12}}}}$
$10_2^{3\times 10.(3{,}000{,}000{,}003)}$	$10 \uparrow\uparrow (3 \times 10 \uparrow 3{,}000{,}000{,}003)$

Let's apply the new Al-Ossmi's free arrows notation in case of $\boldsymbol{a_n^{b.c}}$,to describe such the number:$a^{\left(a^{a^c}\right)}$. The given number is a power tower with the base *a* and height 3 exponents, with the topmost exponent being *c*. From the notation definition, the height or number of iterations in the power tower is *b*, and *n* is the level or number of arrows (exponentiation depth).

Example and interpretation: $\boldsymbol{a}_2^{(\boldsymbol{b}\ copies\ of\ a)^{(c)}} = \boldsymbol{a}_2^{(\boldsymbol{b}\ copies\ of\ a).c}$.

Base $(a) = 10$

Height **(*b*)**: The height of the power tower here includes the topmost exponent and any additional exponents as iterations; $\left(a^{a^c}\right)$:

Since the topmost exponent is $\mathbf{10}^{\left(\mathbf{10}^{\mathbf{10}^{12}}\right)}$, we adjust the height to reflect this deep nesting. Level (n): Since this is straightforward exponentiation (second level of up-arrow), $n = 2$.

We describe it in a notation reflecting $\left(3^{(12)}\right)$ height, combining the depth and extending beyond simple iteration count. If we consider it as an iteration extending beyond the simple height, we express it as: $10^{10^{10^{12}}} = 10_2^{3^{(12)}} = 10_2^{3.12}$,

This flexibility of Al-Ossmi's notation with forms; $\boldsymbol{a_n^{b.c}}$ or $\boldsymbol{a_n^{b.c.d.e.f.g.h}}$ allows for a compact and structured way to represent extremely large numbers. These $(b.c.d.e)$ are components to describe complex and extended depth and extend the depth further, representing additional layers of power towers or nested operations. By accurately reflecting the base, height (with nuanced depth like $b.c.d,$ indicating an extended height in the power tower), and level of exponentiation, it simplifies understanding and interpreting large numbers. The use of multiple components separated by dots allows for a detailed and nuanced representation of the structure of the large number. This notation captures the nuances of the structure of these numbers, including the base, height, depth, and level of operations, making it easier to understand and work with extremely large values.

The original estimate is then when $(n > 1)$, the value of *b* indicates to the tetration of the base, a. More precisely, the examples:

$$\mathbf{2_1^3} = 2 \uparrow 3 = \mathbf{2^3} = 8\,,$$

$$\text{whereas; } \mathbf{2_2^3} = 2 \uparrow\uparrow 3 = 2 \uparrow (2 \uparrow 2) = \mathbf{2}^{(2^2)} = \mathbf{16}$$

note that; $\mathbf{2_3^3} = 2 \uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow \big(2 \uparrow\uparrow (2 \uparrow 2 \uparrow 2)\big) = 2 \uparrow\uparrow \left(2 \uparrow\uparrow \left(\mathbf{2^{2^2}}\right)\right)$

Therefore, compared with notations such as Conway's chained and Knuth's, the value of *n* is related to the number of arrows, while it is by Al-Ossmi's notation indicates that we deal with a tetration process, thus value of *n* in Al-Ossmi's does not help to determine the exact value of the number. Al-Ossmi's arrow-free notation easily helps to write out extremely large power towers, as it is listed in **Table 2** and **3**.

Table 2: Systems of key Notations for Arithmetic Operators.

Arithmetic	Standard	Ackermann's	Knuth's	Conway's	Al-Ossmi's
Exponentiation	a^b	ack(a,b,2)	$a \uparrow b$	$a \to b \to 1$	$\boldsymbol{a^b}$
Tetration	${}^b a$	ack(a,b,3)	$a \uparrow\uparrow b$	$a \to b \to 2$	$\boldsymbol{a_2^b}$
Pentation	b^a	ack(a,b,4)	$a \uparrow\uparrow\uparrow b$	$a \to b \to 3$	$\boldsymbol{a_3^b}$
Hexation	-	ack(a,b,5)	$a \uparrow\uparrow\uparrow\uparrow b$	$a \to b \to 4$	$\boldsymbol{a_4^b}$
Fundamental rule	-	ack(a,b,n)	$a \uparrow^n b$	$a \to b \to n$	$\boldsymbol{a_n^b}$

Where:
a, *b*, *n* are positive integers, hence:
a is the base number,
b copies of *a*,
n is the arrow number.

Table 3: Al-Ossmi's free arrows notation of a set of extremely huge numbers in form of titration exponential express.

Number Name	*Exponential Notation*	*Al-Ossmi's notation*
Skewes number	$10^{10^{10^{34}}}$	$10_2^{3.34}$
Pentalogue	$10^{10^{10^{10^{10}}}}$	10_2^5
Millyllion	$10^{2^{1002}}$	$10_1^{2.1002}$
Gigillion	$10^{3\times10^{3,000,000,000+3}}$	$10_1^{3\times10^{3,000,000,000+3}}$
Ecetonplex	$10^{10^{303}}$	$10_2^{2.303}$
Heskironduplex	$10^{10^{10^{600}}}$	$10_2^{3.600}$
Googolduplexichime	$10^{10^{10^{1000}}}$	$10_2^{3.1000}$
Guppyduplexitoll	$10^{10^{10^{2000}}}$	$10_2^{3.2000}$
Googolduplexibell	$10^{10^{10^{5000}}}$	$10_2^{3.5000}$
Millinillion	10^{3003}	10_1^{3003}
Millinillinillion	$10^{3000003}$	$10_1^{3000003}$
Hepta-taxis	$10 \uparrow\uparrow\uparrow 7$	10_3^7
Hexa-taxis	$10 \uparrow\uparrow\uparrow 6$	10_3^6
Penta-taxis	$10 \uparrow\uparrow\uparrow 5$	10_3^5
Boogafive	$5 \uparrow\uparrow\uparrow 5$	5_3^5
Tetra-taxis	$5 \uparrow\uparrow\uparrow 4$	5_3^4
Gigaexpofaxul	$10 \uparrow\uparrow\uparrow (5+98)$	$10_3^{(5+98)}$
Two	$2 \uparrow 1$	$2_1^1 = 2^1 = 2$

5. Conclusions

In this paper, we introduce a novel notation for expressing extremely large numbers, named Al-Ossmi's notation after its creator. This notation aims to compactly represent large numbers by providing a clear structure that shows the base, the level of iteration, and the depth of the operation. By doing so, it offers an efficient and unambiguous method for handling vast numerical values, making it a valuable tool for mathematicians and computer scientists.

Al-Ossmi's arrow-free notation is defined as; a_n^b , $a_n^{b.c}$, or $a_n^{b.c.d.e}$, where:

- a: The base number.
- *b,c,d,e etc.*: The number of iterations or the height of the power tower.
- n: The level of operation or the number of arrows in Knuth's notation.

The original estimate is then this notation can be extended to include more complex structures, such as a_n^b , $a_n^{b.c}$, or $a_n^{b.c.d.e.f.g}$, to represent additional levels of nested operations, where d and c, are variables. Al-Ossmi's arrow-free notation simplifies the representation of very large numbers by using a compact form that corresponds to $(a \uparrow^n b)$ in Knuth's notation and $a \rightarrow b \rightarrow n$ in the Conway's chained arrow notation. It combines these notations into a concise and easily readable format, reducing the complexity and length of numerical expressions. This notation is more standardized and better recognized within the mathematical community, making it effective for communicating and working with extremely large numbers. It is less cumbersome than writing multiple up-arrows or chaining arrows and is easy to write and understand once the rules are clear. To facilitate the adoption of Al-Ossmi's notation, detailed documentation and examples are provided. This includes practical applications in various fields such as physics, astronomy, and large number theory, where extremely large numbers are common. Al-Ossmi's arrow-free notation utilities in different fields, in physics or astronomy, this notation can simplify expressions and calculations involving vast quantities. In combinatorial mathematics or proofs involving large number theory, it provides clarity and precision.

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